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On three kinds of lower-limit oscillation tests for linear first-order differential equations with several delays[1]

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Abstract. We consider linear non-autonomous delay differential equations of first order, and study effective conditions guaranteeing that all solutions of an equation oscillate, in the form of an estimate for the lower limit at infinity of a functional of coefficients and delays of the equation. Such conditions generalize the well-known Myshkis oscillation test for a linear equation with a single delay, in the form of an inequality with $1/e$ in its right-hand side. We consider three different approaches to obtaining such conditions for equations with several concentrated delays, and compare these approaches with each other. We present several new examples in order to show the effectiveness of the approaches and advantages of each of them over the others.

Keywords: delay differential equations, oscillation, explicit test.

1. Introduction

Linear first-order delay differential equations, in contrast to linear first-order ordinary differential equations, can have oscillating solutions. In particular, due to this property of solutions, equations with aftereffect serve as good mathematical models for describing a number of processes in biology, chemistry, and the theory of automatic control.

The problem of oscillation conditions for first-order differential equations was first investigated by Myshkis [15]. For 20 years after his pioneering works the problem did not attract researchers, since the directions of systematic research of equations with aftereffect in that time were mainly determined by the desire to transfer known results of the theory of ordinary differential equations to delay equations. Several papers on the oscillation of solutions to first-order equations were published in the 1970s. They led to the Koplatadze—Chanturiya theorem [10] on conditions for oscillation of all solutions of a linear non-autonomous equation. This theorem generalized results by Myshkis and Ladas [12]. Over the past 40 years, the number of papers devoted to effective oscillation conditions for first-order delay equations has been steadily growing. However, despite the fact that some of recent papers of this kind contain reviews of known results, there

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is no complete portrait of the phenomenon. This article is devoted to some of the most significant achievements in the study of effective oscillation tests. Namely, we consider sufficient conditions for the oscillation of all solutions to linear non-autonomous equations of stable type with several delays. The conditions should be explicitly expressed in terms of parameters of a given equation.

The paper is organized as follows. In the second section we specify some questions that arose when studying the oscillation problem for linear first-order equations. In the third section we consider three fruitful approaches to generalizing the Myshkis lower-limit oscillation test in the form of inequality with $1/e$ in its right-hand side. Two of the approaches are generalizations of the integral oscillation condition obtained by Koplatadze and Chanturiya, and the third one is the Hunt—Yorke theorem. In the fourth section we present several new examples in order to compare the domains of applicability of the approaches represented in the third section.

2. Lower-limit oscillation tests

Definition 1. We say that a continuous on the semiaxis $\mathbb{R}_+ \equiv [0,+\infty)$ function *oscillates*, if it has an unbounded from the right sequence of zeros.

Definition 2. We say that a differential equation is *oscillatory*, if its every solution oscillates.

The following criterion is well known: the autonomous delay equation

$$\dot{x}(t) + ax(t-r) = 0$$

is oscillatory if and only if $ar > 1/e$. For the oscillation of solutions to non-autonomous equations we can only talk about sufficient conditions. Naturally, we suppose that the best of oscillation tests are those for which both the essentiality of all imposed requirements and the sharpness of all constants in inequalities are established. The above test was first transferred to the case of non-autonomous equations by Myshkis [15]. Below we consider oscillation tests generalizing these results.

2.1. The Koplatadze—Chanturiya theorem

Assume that the functions a and h are continuous on $\mathbb{R}_+$, and let $h(t) \le t$.

Consider the linear non-autonomous equation with a single delay

$$\dot{x}(t) + a(t)x(h(t)) = 0, \quad t \in \mathbb{R}_+. \tag{1}$$

We say that a continuously differentiable function $x\colon \mathbb{R}_+ \to \mathbb{R}$ is a *solution* to equation (1), if x satisfies (1) everywhere provided that for $\xi \le 0$ we have $x(\xi) = \varphi(\xi)$ for some *initial function* φ. It is easily seen that for any continuous function φ there is a unique solution to (1).

Theorem 1 ([10]). *If* $a(t) \geq 0$, $\lim\limits_{t\to\infty} h(t) = +\infty$ *and*

$$\liminf_{t\to+\infty} \int\limits_{h(t)}^{t} a(s)\,ds > 1/e, \tag{2}$$

then equation (1) *is oscillatory.*

Let us consider the hypotheses of Theorem 1. The condition $a_k(t) \geq 0$ means that equations under consideration are those of *stable type*. The condition $\lim\limits_{t\to\infty} h(t) = +\infty$ was for the first time introduced by Myshkis as a weakening of the condition of the boundedness of delay. Myshkis showed the essentiality of this condition, therefore below we will call it and its analogues for equations of more general form *the Myshkis condition*. The last few decades, as a rule, researchers of asymptotics of delay differential equations have imposed it without any explanation. The main condition of Theorem 1 is the inequality (2). The case of the autonomous equation shows that the constant $1/e$ in (2) is sharp, i.e. it cannot be diminished. Moreover, the strict inequality (2) cannot be replaced by the nonstrict one. However, one can refine the left-hand side of the inequality and make oscillation conditions to be applicable to wider classes of equations. The number of papers devoted to refinements and generalizations of Theorem 1 is growing every year.

2.2. The equation with several delays

One of the main directions of generalizations of Koplatadze—Chanturiya theorem is the transfer of the result to equations with several delays.

Consider the equation

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(h_k(t)) = 0, \quad t \in \mathbb{R}_+, \tag{3}$$

where it is assumed that the functions a_k and h_k are piecewise continuous, and $h_k(t) \leq t$, $k = \overline{1,n}$. By analogy with equation (1), we suppose that a *solution* to equation (3) is a continuously differentiable function $x\colon \mathbb{R}_+ \to \mathbb{R}$ that satisfies (3) for some continuous initial function $\varphi\colon (-\infty, 0] \to \mathbb{R}_+$.

Futher, we everywhere suppose that $a_k(t) \geq 0$, that is equation (3) is of stable type, and that the Myshkis condition $\lim\limits_{t\to\infty} h_k(t) = +\infty$ is fulfilled, $k = \overline{1,m}$.

Since equation (1) is a special case of (3), there is a natural desire to generalize Theorem 1 to equation (3). First, consider the autonomous equation

$$\dot{x}(t) + \sum_{k=1}^{m} a_k x(t - r_k) = 0. \tag{4}$$

Theorem 2 ([13]). *For equation* (4) *to be oscillatory it is sufficient that*

$$\sum_{k=1}^{m} a_k r_k > 1/e. \tag{5}$$

Now, by virtue of Theorems 1 and 2, it is natural to set the following

Problem. Is it true that if

$$\liminf_{t\to+\infty} \sum_{k=1}^{m} \int\limits_{h_k(t)}^{t} a_k(s)\,ds > 1/e, \tag{6}$$

then equation (3) is oscillatory?

In fact, Problem was not explicitly posed in literature; nevertheless, since 1982 many attempts have been made to bring sufficient oscillation conditions for equation (3) as close as possible to the inequality (6). However, the answer to the question in Problem is negative, which was shown by a counterexample in the recent paper [6].

The simplest way to generalize Theorem 1 to the case of equation (3) is to take an interval of integration small enough to be contained in all the intervals $[h_k(t), t]$, $k = \overline{1, m}$. Results of this kind are deduced from the following special case of the Myshkis theorem on the comparison of solutions [16, p. 177].

Theorem 3. *If equation* (3) *is oscillatory and* $\tilde{a}_k(t) \geq a(t) \geq 0$, $\tilde{h}_k(t) \leq h(t)$, $k = \overline{1, m}$, *then the equation*

$$\dot{x}(t) + \sum_{k=1}^{m} \tilde{a}_k(t) x(\tilde{h}_k(t)) = 0, \quad t \in \mathbb{R}_+,$$

is also oscillatory.

The following fact is obtained immediately from Theorems 1 and 3.

Denote $g(t) = \max\limits_{k\in\{1,\ldots,m\}} \sup\limits_{s\in[0,t]} h_k(s)$.

Corollary 1. *If*

$$\liminf_{t\to\infty} \int\limits_{g(t)}^{t} \sum_{k=1}^{m} a_k(s)\,ds > 1/e,$$

then equation (3) *is oscillatory.*

Remark 1. The result equivalent to Corollary 1 was apparently first set in [8]. Unfortunately, contemporary authors do not read works by Myshkis, so they have to derive consequences of his results from scratch.

Corollary 1 has an evident weakness: the effect of relatively large delays on the oscillation is not taken into account; e. g., if $h_k(t) = t$ for some k, then the result completely loses force.

3. Three approaches to testing oscillation

In this section we consider the most effective of the known approaches to generalizing Theorems 1 and 2 to the case of equation (3).

3.1. Iterative method

Consider a fruitful approach, which allows to refine the hypotheses in Corollary 1 and take into account the effect of all delays on the oscillation of equation (3). It consists in constructing an iterative sequence of statements such that each of them refines oscillation conditions presented in the previous one. This approach goes back to paper [11] by Koplatadze and Kvinikadze. In works of other authors it has been developed only since 2016.

Denote $P_0(t,s) = 1$ and

$$P_n(t,s) = \exp\left\{ \int_s^t \sum_{k=1}^m a_k(\zeta) P_{n-1}(\zeta, h_k(\zeta))\, d\zeta \right\}, \quad n \in \mathbb{N}. \tag{7}$$

Lemma 1 ([3])**.** *If a solution x of* (3) *is positive for $t \geq t_0$, then there exists $t_1 \geq t_0$ such that for all $n \in \mathbb{N}$ and $t \geq s \geq t_1$ we have*

$$x(t) P_n(t,s) \leq x(s).$$

Theorem 4 ([5])**.** *If for some $n \in \mathbb{N}$ the inequality*

$$\liminf_{t\to\infty} \int_{g(t)}^t \sum_{k=1}^m a_k(s) P_n(g(s), h_k(s))\, ds > 1/e, \tag{8}$$

is true, then equation (3) *is oscillatory.*

If one replaces $g(s)$ by $g(t)$ in (8), then Theorem 4 turns into Theorem 6 from [3], which is erroneous, see [5]. There are corrected versions of [3] published at arXiv.org in 2017 and 2019.

It is possible to increase the left-hand side in the inequality (8) at the cost of increasing the number of integrals in (8) or (7). Such a refinement of Theorem 4 is presented, e. g., in [4]. It is curious that the incorrect Theorem 6 from [3] is quoted in [4] as well as in some other papers, results of which would be useless if this theorem were true. However, there are many absurdities in works of recent years on the topic under discussion.

3.2. Change of integration sets

Consider another recent approach to the refinement and generalization of Theorem 1. For $k = \overline{1,m}$ define families of sets

$$E_k(t) = \{s \geq t : h_k(s) < t\}, \quad t \in \mathbb{R}_+.$$

For what follows, we extend the class of equations (3). Assume that the functions a_k are locally integrable, the functions h_k are Lebesgue measurable, and $h_k(t) \leq t$ for almost all $t \in \mathbb{R}_+$, $k = \overline{1,m}$. Below we mean that a *solution* to equation (3) is a locally absolutely continuous function $x\colon \mathbb{R}_+ \to \mathbb{R}$ that satisfies (3) almost everywhere on $\mathbb{R}_+$. It is easily seen (see, e. g., [1, Ch. 1]) that for any given bounded Borel initial function $\varphi\colon (-\infty, 0] \to \mathbb{R}$ provided that $x(\xi) = \varphi(\xi)$, $\xi \leq 0$, and *initial value* $x(0)$ there exists a unique solution to equation (3).

We set $a_k(t) \geq 0$ for almost all $t \in \mathbb{R}_+$, and $h_k(t) \to \infty$ as $t \to \infty$ essentially, that is for t defined in some $M \subset \mathbb{R}_+$ such that $\mu(\mathbb{R} \setminus M) = 0$.

Theorem 5 ([7]). *If*

$$\liminf_{t\to\infty} \sum_{k=1}^{m} \int_{E_k(t)} a_k(s)\, ds > 1/e,$$

then equation (3) *is oscillatory.*

Theorem 5 is significantly stronger than Theorem 1 even in the case $m = 1$, see [7].

Corollary 2. *If the functions h_k are continuous and strictly increasing to infinity on $\mathbb{R}_+$, $k = \overline{1,m}$, and*

$$\liminf_{t\to\infty} \sum_{k=1}^{m} \int_{t}^{h_k^{-1}(t)} a_k(s)\, ds > 1/e,$$

then equation (3) *is oscillatory.*

Corollary 3 ([14]). *If $h_k(t) = t - r_k$, where $r_k = const > 0$, $k = \overline{1,m}$, and*

$$\liminf_{t\to\infty} \sum_{k=1}^{m} \int_{t}^{t+r_k} a_k(s)\, ds > 1/e, \tag{9}$$

then equation (3) *is oscillatory.*

Note that if $m > 1$, then, in general,

$$\liminf_{t\to\infty} \sum_{k=1}^{m} \int_{t}^{t+r_k} a_k(s)\, ds \neq \liminf_{t\to\infty} \sum_{k=1}^{m} \int_{t-r_k}^{t} a_k(s)\, ds.$$

If one replaces the intervals of integration $[t, t + r_k]$ by $[t - r_k, t]$ in (9), then (9) turns into a special case of (6), and Corollary 3 turns into an erroneous statement, see [6].

However, in some papers Theorem 5 from [14], which is the same as Corollary 3, is quoted with this incorrect replacement.

There is a question, is it possible to apply to Theorem 5 an iterative approach similar to that described in subsection 3.1. In this regard, we note that iterative refinements of Theorem 1 and Corollary 1 take place due to the fact that these propositions are insensitive to the non-monotonicity of the functions h and h_k. It is not the case for Theorem 5, which takes into account all the values of all delays. If an iterative generalization of Theorem 5 is possible, then it requires a more subtle approach.

3.3. The Hunt—Yorke theorem

The following result stands apart among lower-limit oscillation conditions for (3).

Theorem 6 ([9]). *If the functions a_k and h_k are continuous, the functions $r_k(t) = t - h_k(t)$ are bounded, $a_k(t), r_k(t) > 0$, $k = \overline{1,m}$, and*

$$\liminf_{t\to+\infty} \sum_{k=1}^{m} a_k(t) r_k(t) > \frac{1}{e},$$

then equation (3) *is oscillatory.*

Theorem 6, like Theorems 4 and 5, generalize Theorem 2. The conditions in Theorem 6 can be called *pointwise*, in contrast to integral conditions like those of Theorems 1, 4 and 5. Surprisingly, Theorem 6, which is quoted in a large number of works, has in fact not been compared in strength with integral conditions of oscillations.

Note that one of the hypotheses of Theorem 6 assumes that delays are limited. This condition is essentially used in the proof given by the authors in [9]. The natural question of whether it is possible to replace the condition of limited delays by the Myshkis condition, that is, apparently, has not yet been posed in literature. We do not answer this question here, but we can conjecture that this weakening of the assumptions of Theorem 6 is correct.

4. Comparison of oscillation tests

In this section we compare with each other the classes of equations of the form (3) with parameters satisfying the hypotheses of Theorems 4, 5 and 6. We call these classes the *domains of applicability* of the theorems. We give a number of examples showing that for each of Theorems 4, 5 and 6 its domain of applicability is not included into that of any of the other two theorems even in the case of a single delay. The results of this section are valid regardless of whether we consider equation (3) under the assumptions of subsection 2.2 or those of subsection 3.2.

The easiest case of our task is to show that the domain of applicability of Theorem 5 is not included into any of those of Theorems 4 and 6. For this end, it is sufficient to consider an equation with periodically vanishing delay.

Consider the equation

$$\dot{x}(t) + a(t)x(t - r(t)) = 0, \quad t \in \mathbb{R}_+, \tag{10}$$

and denote $g(t) = \sup\limits_{s\in[0,t]} (s - r(s))$.

Example 1. Set in equation (10) $a(t) \equiv 1$ and

$$r(t) = \begin{cases} 1, & t \in [n, n + 1/2); \\ 0, & t \in [n + 1/2, n + 1); \end{cases} \qquad n = 0, 1, 2, \dots$$

We have

$$\liminf_{t\to\infty} \int_{E(t)} a(s)\,ds = \int_n^{n+1/2} a(s)\,ds = 1/2 > 1/e;$$

$$\liminf_{t\to\infty} a(t)r(t) = \liminf_{t\to\infty} \int_{g(t)}^t a(s)P_n(g(s), h(s))\,ds = 0 < 1/e.$$

Thus, the equation is oscillatory by Theorem 5, but it cannot be established by any of Theorems 4 and 6.

The following example shows that the domain of applicability of Theorem 6 is not included into any of those of Theorems 4 and 5.

Example 2. Consider equation (10), where for $n = 0, 1, 2, \dots$ set

$$a(t) = \begin{cases} 1/3, & t \in [3n, 3n + 2); \\ 2/3, & t \in [3n + 2, 3(n + 1)); \end{cases} \qquad r(t) = \begin{cases} 2, & t \in [3n, 3n + 2); \\ 1, & t \in [3n + 2, 3(n + 1)). \end{cases}$$

We have

$$\liminf_{t\to\infty} a(t)r(t) = 2/3 > 1/e;$$

however,

$$\liminf_{t\to\infty} \int_{g(t)}^t a(s)P_n(g(s), h(s))\,ds = \int_{g(3n+2)}^{3n+2} a(s)P_n(g(s), h(s))\,ds$$
$$= \int_{3n+1}^{3n+2} a(s)\,ds = 1/3 < 1/e,$$

and

$$\liminf_{t\to\infty} \int_{E(t)} a(s)\,ds = \int_{E(3n+1)} a(s)\,ds = \int_{3n+1}^{3n+2} a(s)\,ds = 1/3 < 1/e.$$

Thus, the equation is oscillatory by Theorem 6, but it cannot be established by any of Theorems 4 and 5.

At last, the following example shows that the domain of applicability of Theorem 4 is not included into any of those of Theorems 5 and 6.

Example 3. Consider equation (10), where $a(t) = 1/e$, and for $n = 0, 1, 2, \ldots$ we put

$$h(t) = t - r(t) = \begin{cases} t-2, & t \in [4n, 4n+2); \\ 4n-1, & t \in [4n+2, 4n+3); \\ 4n+2, & t \in [4n+3, 4(n+1)). \end{cases}$$

One can see that

$$\liminf_{t\to+\infty} \int_{E(t)} a(s)\,ds = \int_{E(4n+2)} a(s)\,ds = \int_{4n+2}^{4n+3} a(s)\,ds = 1/e,$$

and

$$\liminf_{t\to+\infty} a(t)r(t) = a(4n+3)r(4n+3) = (1/e)\cdot 1 = 1/e.$$

However,

$$\liminf_{t\to+\infty} \int_{g(t)}^{t} a(s)P_1(g(s),h(s))\,ds = \liminf_{t\to+\infty} \int_{g(t)}^{t} a(s)\exp\left(\int_{h(s)}^{g(s)} a(\tau)\,d\tau\right) ds$$
$$= \int_{4n+2}^{4n+3} a(s)\exp\left(\int_{h(s)}^{g(s)} a(\tau)\,d\tau\right) ds > \int_{4n+2}^{4n+3} a(s)\,ds = 1/e.$$

Thus, equation (10) is oscillatory, which is established by Theorem 4 and cannot be established by any of Theorems 5 and 6.

5. Conclusion

We have examined a class of sufficient conditions for the oscillation of all solutions to linear non-autonomous differential equations of first order with several delays, which is called a class of lower-limit oscillation tests. For our analysis, we have chosen three approaches to constructing such tests, which we consider to be the most effective.

As to the generalization and refinement of the represented results, we can say the following. The iterative oscillation tests allow refinement at the cost of a significant complication of formulas (7) and (8) [4]. The Hunt—York approach was tranferred to the case of equations with distributed delay in [2]. Generalizations of Theorem 5 are currently unknown.

There are other approaches to oscillation testing. In particular, it should be noted that oscillation conditions can be formulated in terms of estimates for not the lower, but the upper limit of a functional of equation parameters. Moreover, the iterative approach was first developed in [11] specifically for this kind of oscillation tests.

References

1. *Azbelev, N. V., Maksimov, V. P., Rakhmatullina, L. F.* Introduction to the theory of linear functional-differential equations. Transl. from the Russian. Advanced Series in Mathematical Science and Engineering, 3. Atlanta: World Federation Publishers Company, 1995. Zbl 0867.34051
2. *Berezansky, L., Braverman, E.* On oscillation of equations with distributed delay. Z. Anal. Anwend. 20, No. 2, 489–504 (2001). Zbl 0995.34059
3. *Braverman, E., Chatzarakis, G. E., Stavroulakis, I. P.* Iterative oscillation tests for differential equations with several non-monotone arguments. Adv. Difference Equ. 2016, Paper No. 87, 18 p. (2016). Zbl 1348.34121
4. *Chatzarakis, G. E., Dix, J. G.* Oscillations of nonlinear differential equations with several deviating arguments. Electron. J. Qual. Theory Differ. Equ. 2018, Paper No. 39, 12 p. (2018). Zbl 1413.34214
5. *Chudinov, K. M., Malygina, V. V.* On oscillation of linear differential equations with several delays, Bulletin of Perm University. Mathematics. Mechanics. Computer science. No. 4, 11-18 (2017). (in Russian)
6. *Chudinov, K. M.* On exact sufficient oscillation conditions for solutions of linear differential and difference equations of the first order with aftereffect. Russian Math. (Iz. VUZ) 62, 79–84 (2018). Zbl 1403.34049
7. *Chudinov, K. M.* On the conditions for oscillation of the solutions to differential equations with aftereffect and generalization of the Koplatadze-Chanturiya theorem. Sib. Math. J. 61, 178–186 (2020).
8. *Fukagai, N., Kusano, T.* Oscillation theory of first order functional-differential equations with deviating arguments. Ann. Mat. Pura Appl. 136, No. 4, 95-117 (1984). Zbl 0552.34062
9. *Hunt, B. R., Yorke, J. A.* When all solutions of $x' = -\sum q_i(t)x(t - T_i(t))$ oscillate. J. Differential Equations 53, No. 2, 139–145 (1984). Zbl 0571.34057
10. *Koplatadze, R. G., Chanturiya, T. A.* Oscillating and monotone solutions of first-order differential equations with deviating argument. Differentsial'nye Uravneniya 18, No. 8, 1463–1465 (1982). (in Russian) Zbl 0496.34044
11. *Koplatadze, R., Kvinikadze, G.* On the oscillation of solutions of first-order delay differential inequalities and equations. Georgian Math. J. 1, No. 6, 675–685 (1994).
12. *Ladas G.* Sharp conditions for oscillations caused by delays. Applicable Anal. 9, No. 2, 93–98 (1979). Zbl 0407.34055
13. *Ladas, G., Stavroulakis, I. P.* Oscillations caused by several retarded and advanced arguments. J. Differ. Equations 44, 134-152 (1982). Zbl 0452.34058
14. *Li, B.* Oscillation of first order delay differential equations. Proc. Amer. Math. Soc. 124, No. 12, 3729–3737 (1996). Zbl 0865.34057
15. *Myshkis, A. D.* On solutions of linear homogeneous differential equations of the first order of stable type with a retarded argument. Mat. Sb., N. Ser. 28(70), 641-658 (1951). (in Russian) Zbl 0042.32802
16. *Myshkis, A. D.* Linear differential equations with a retarded argument. Moscow: Nauka, 1972. (in Russian) Zbl 0261.34040

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