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Dynamics of topological flows and homeomorphisms with a finite hyperbolic chain-recurrent set on n-manifolds[1]

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Abstract. Starting from dimension 4, so-called non-smoothed manifolds, manifolds that do not allow triangulation and other obstacles that prevent the use of the technique of smooth manifolds for the study of multidimensional manifolds appear. In addition, all methods for studying smooth dynamical systems on multidimensional manifolds are based on the approximation of all subsets by piecewise linear or topological objects. In this regard, the idea of consideration of dynamical systems on multidimensional manifolds that do not use the concept of smoothness in their definition is completely natural. So homeomorphisms and topological Morse-Smale flows, which are also firmly connected with the topology of the ambient manifold, as well as their smooth analogues, have already entered into scientific usage. In the present paper we investigate general dynamical properties of homeomorphisms and topological flows with a finite hyperbolic chain recurrent set.

Keywords: topological flow, chain-recurrent set, hyperbolic set

1. Introduction and formulation of results

Let M^n be a closed n-dimensional manifold with metric d. *A topological flow* on M^n is a family of homeomorphisms $f^t\colon M^n \to M^n$ that continuously depends on $t \in \mathbf{R}$ and satisfies the following conditions:

1) $f^0(x) = x$ for any point $x \in M^n$;

2) $f^t(f^s(x)) = f^{t+s}(x)$ for any $s, t \in \mathbf{R}$, $x \in M^n$.

The trajectory or the orbit of a point $x \in M^n$ is the set $\mathcal{O}_x = \{f^t(x), t \in \mathbf{R}(\mathbf{Z})\}$. It is believed that the trajectories of the flow (homeomorphism) are oriented in accordance with an increase in the parameter t. Any two trajectories of a dynamical system either coincide or do not intersect, therefore, the phase space is represented as a union of pairwise disjoint trajectories. There are three types of trajectories:

1) *fixed point* $\mathcal{O}_x = \{x\}$;

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2) *periodic trajectory (orbit)* $\mathcal{O}_x$ for which there exists a number $per(x) > 0$ ($per(x) \in \mathbb{N}$) such that $f^{per(x)}(x) = x$, but $f^t(x) \neq x$ for all real (natural) numbers $0 < t < per(x)$. The number $per(x)$ is called *period of a periodic orbit* and does not depend on the choice of a point in orbit;

3) *regular trajectory* $\mathcal{O}_x$ — a trajectory that is not a fixed point or a periodic orbit.

To characterize the wandering of the trajectories of a dynamical system, the concept of chain recurrence is traditionally used.

The *ε-chain of length* T connecting the point x with the point y for the flow f^t is called a sequence of points $x = x_0, \dots, x_n = y$ for which there exists a sequence of times $t_1, \dots, t_n$ such that $d(f^{t_i}(x_{i-1}), x_i) < \varepsilon$, $t_i \geq 1$ for $1 \leq i \leq n$ and $t_1 + \dots + t_n = T$.

The *ε-chain of length* n connecting the point x with the point y for a homeomorphism f is called a sequence of points $x = x_0, \dots, x_n = y$, such that $d(f(x_{i-1}), x_i) < \varepsilon$ for $1 \leq i \leq n$.

A point $x \in M^n$ is said to be *chain recurrent* for the flow f^t (cascade f), if for any $\varepsilon > 0$ there is $T(n)$, which depends on $\varepsilon > 0$, and there is an ε-chain of the length $T(n)$ from the point x to itself. The set of chain recurrent points of f^t (f) is called the *chain recurrent set* of f^t (f) denoted by $\mathcal{R}_{f^t}$ $(\mathcal{R}_f)$ and its connected components are called *chain components.* The set $\mathcal{R}_{f^t}$ $(\mathcal{R}_f)$ is f^t (f) - invariant, that is, it consists of the orbits of the flow (homeomorphism) f^t (f), which are called *chain recurrent.* It is obvious that fixed points and periodic orbits are chain recurrent.

As a model behavior of flow (homeomorphism) in a neighborhood of a fixed point, we consider a linear flow (homeomorphism) $a^t_\lambda : \mathbf{R}^n \to \mathbf{R}^n$ $(a_\lambda : \mathbf{R}^n \to \mathbf{R}^n), \lambda \in \{0, 1, ..., n\}$ of the following form:

$$a^t_\lambda(x_1, ..., x_\lambda, x_{\lambda+1}, ..., x_n) = (2^t x_1, ..., 2^t x_\lambda, 2^{-t} x_{\lambda+1}, ..., 2^{-t} x_n)$$

$$(a_\lambda(x_1, ..., x_\lambda, x_{\lambda+1}, ..., x_n) = (\pm 2x_1, 2x_2, ..., 2x_\lambda, \pm 2^{-1} x_{\lambda+1}, 2^{-1} x_{\lambda+2}, ..., 2^{-1} x_n)).$$

A fixed point p of a flow (homeomorphism) f^t (f) is called *is topologically hyperbolic* if there exists a neighborhood $U_p \subset M^n$, a number $\lambda \in \{0, 1, ..., n\}$ and a homeomorphism $h_p : U_p \to \mathbf{R}^n$ such that $h_p f^t|_{U_p} = a^t_{\lambda_p} h_p|_{U_p}$ $(h_p f|_{U_p} = a_{\lambda_p} h_p|_{U_p})$ whenever the left and right sides are defined. Let

$$E^s_\lambda = \{(x_1, ..., x_n) \in \mathbf{R}^n : x_1 = \dots = x_\lambda = 0\},$$

$$E^u_\lambda = \{(x_1, ..., x_n) \in \mathbf{R}^n : x_{\lambda+1} = \dots = x_n = 0\}.$$

The number λ_p is called the *index* of the fixed hyperbolic point p. A point of indexes n and 0 will be called *source* and *sink*, respectively; any point p such that $\lambda_p \in \{1, \cdots, n-1\}$ is called *saddle.* For a topologically hyperbolic fixed point p of the flow (homeomorphism) f^t (f) the sets $h^{-1}_p(E^s_{\lambda_p}), h^{-1}_p(E^u_{\lambda_p})$ are called *local stable, unstable manifolds.*

The sets

$$W^s_p = \bigcup_{t \in \mathbf{R}} f^t(h^{-1}_p(E^s_{\lambda_p})),\ W^u_p = \bigcup_{t \in \mathbf{R}} f^t(h^{-1}_p(E^u_{\lambda_p}))$$

is called *stable and unstable invariant manifolds of the point p*.

If p is a periodic point of a period k for diffeomorphism f then its invariant manifolds and the index are defined as for fixed point $f^k(p)$ with respect to the homeomorphism f^k. The number $\lambda_{\mathcal{O}_p}$ which equals λ_p is called *index of the orbit* $\mathcal{O}_p$ of the periodic point p.

Statement 1. *The unstable W^u_p and the stable W^s_p manifolds of the hyperbolic fixed point p are independent of the choice of the local homeomorphism h_p and are defined in topological terms as follows: $W^u_p = \{y \in M^n : \lim\limits_{t\to+\infty} f^{-t}(y) = p\}$ u $W^s_p = \{y \in M^n : \lim\limits_{t\to+\infty} f^{t}(y) = p\}$.*

It follows from Statement 1 that $W^u_p \cap W^u_q = \emptyset$ and $W^s_p \cap W^s_q = \emptyset$ for any different hyperbolic points p, q.

Denote by G a class of homeomorphisms and topological flows given on M^n with a finite hyperbolic chain recurrent set.

Let $\mathcal{F} \in G$. The dynamics of systems of this class are close in their properties to gradient-like systems (see, for example, [4], [2]). Namely, similar to S. Smale's order [5], we introduce a partial order relation on the set of chain-recurrent orbits of the dynamical system $\mathcal{F}$ by the condition

$$\mathcal{O}_i \prec \mathcal{O}_j \iff W^s_{\mathcal{O}_i} \cap W^u_{\mathcal{O}_j} \neq \emptyset,$$

where $\mathcal{O}_i, \mathcal{O}_j$ are orbits from the set $\mathcal{R}_{\mathcal{F}}$ and $W^s_{\mathcal{O}_i} = \bigcup\limits_{p\in\mathcal{O}_i} W^s_p$, $W^u_{\mathcal{O}_i} = \bigcup\limits_{p\in\mathcal{O}_i} W^u_p$.

A m-cycle ($m \geq 1$) is a collection $\mathcal{O}_1, \mathcal{O}_2, \cdots, \mathcal{O}_m$ of pairwise disjoint chain recurrent orbits that satisfy the condition $\mathcal{O}_1 \prec \mathcal{O}_2 \prec \cdots \prec \mathcal{O}_m \prec \mathcal{O}_1$.

Statement 2. *Every dynamical system $\mathcal{F} \in G$ has no cycles.*

Due to Statement 2 the introduced relation can be continued (not uniquely) to a complete order relation, that is for every chain recurrent orbits $\mathcal{O}_i$, $\mathcal{O}_j$ either $\mathcal{O}_i \prec \mathcal{O}_j$, or $\mathcal{O}_j \prec \mathcal{O}_i$. Thus, let us consider the orbits of a dynamical system $\mathcal{F} \in G$ numbered in accordance with the introduced order:

$$O_1 \prec \cdots \prec \mathcal{O}_k.$$

In addition, without loss of generality, we assume that any sink orbit is located in this order below any saddle orbit, and any source orbit is higher than any saddle one.

The main result of the present paper is the following fact.

Theorem 1. *Let $\mathcal{F} \in G$. Then*

(1) $M^n = \bigcup\limits_{i=1}^{k} W^u_{\mathcal{O}_i} = \bigcup\limits_{i=1}^{k} W^s_{\mathcal{O}_i}$;

(2) $W^u_{\mathcal{O}_i}$ $(W^s_{\mathcal{O}_i})$ *s a topological submanifold of M^n, homeomorphic to* $\mathbf{R}^{\lambda_{\mathcal{O}_i}}$ $(\mathbf{R}^{n-\lambda_{\mathcal{O}_i}})$;

(3) $(cl(W^u_{\mathcal{O}_i}) \setminus W^u_{\mathcal{O}_i}) \subset \bigcup\limits_{j=1}^{i-1} W^u_{\mathcal{O}_j}$ $((cl(W^s_{\mathcal{O}_i}) \setminus W^s_{\mathcal{O}_i}) \subset \bigcup\limits_{j=i+1}^{k} W^s_{\mathcal{O}_j})$.

Notice that a similar result for Morse-Smale diffeomorphisms was proved in [3] and for Morse-Smale homeomorphism was proved in [1].

2. Auxilary facts

In this section we prove announced statements.

Proof of Statement 1.

Proof. Let us prove that if $\mathcal{F} \in G$ and p is a fixed point of the system $\mathcal{F}$, then W^u_p and W^s_p are independent of the choice of the local homeomorphism h_p.

Suppose for definiteness that $\mathcal{F}$ is a homeomorphism f (for a flow the proof is similar). Let $\tilde{h}_p : \tilde{U}_p \to \mathbf{R}^n$ be a homeomorphism different from h_p and such that $\tilde{h}_p f|_{\tilde{U}_p} = a_{\tilde{\lambda}_p} \tilde{h}_p|_{\tilde{U}_p}$ whenever the left and right sides are defined. Then in a neighborhood U_O of the original point O in $\mathbb{R}^n$ is well-defined a homeomorphism $h = h_p \tilde{h}_p^{-1}$ which conjugates a_{λ_p} with $a_{\tilde{\lambda}_p}$. As conjugating homeomorphism preserves the invariant manifolds then $\tilde{\lambda}_p = \lambda_p$ and $h(E^s_{\lambda_p}) = E^s_{\lambda_p}$, $h(E^u_{\lambda_p}) = E^u_{\lambda_p}$. Thus, $\tilde{h}_p^{-1}(E^s_{\lambda_p}) = h_p^{-1}(h(E^s_{\lambda_p})) = h_p^{-1}(E^s_{\lambda_p})$. It is the same for $E^u_{\lambda_p}$. □

Proof of Statement 2.

Proof. We will prove that every dynamical system $\mathcal{F} \in G$ has no cycles.

Suppose the contrary: there exists a sequence of orbits $\mathcal{O}_1 \prec \cdots \prec \mathcal{O}_m \prec \mathcal{O}_1$. By construction, any point of the set $\bigcup\limits_{i=1}^{m}(W^s_{\mathcal{O}_i} \cap W^u_{\mathcal{O}_{i+1}})$, where $\mathcal{O}_{m+1} = \mathcal{O}_1$, is a chain recurrent. It immediately contradicts with the finiteness of the chain recurrent set of the system $\mathcal{F}$. □

Statement 3. *Every homeomorphism $f = f^1$, which is the one-time shift of a flow $f^t \in G$ belongs to the class G. Moreover, $\mathcal{R}_{f^t} = \mathcal{R}_f$ and $W^u_p(f^t) = W^u_p(f)$, $W^s_p(f^t) = W^s_p(f)$ for every chain recurrent point p.*

Proof. It immediately follows from the definition of hyperbolic point that $\mathcal{R}_{f^t} \subset \mathcal{R}_f$. Let us show that $\mathcal{R}_f \subset \mathcal{R}_{f^t}$. Let $p \in \mathcal{R}_f$, then there exists a ε-chain of length n connecting the point p with itself. Then the point p will also be a chain recurrence point for the flow f^t, since it has a ε-chain of length $T = n$, and $t_i = 1$ $(i = \overline{1;n})$ connecting the point p with itself.

It follows from the uniqueness of the invariant manifolds of a chain recurrent point, proved in Statement 1, that $W^u_p(f^t) = W^u_p(f)$, $W^s_p(f^t) = W^s_p(f)$ for every chain recurrent point p. □

3. General dynamical properties of systems from class G

In this section, we prove Theorem 1 on the embedding and asymptotic behaviour of invariant manifolds of chain recurrent points of a dynamical system from class G. Due to Statement 3, it is enough to prove the theorem for the case when the dynamical system $\mathcal{F}$ is a homeomorphism f. Moreover we can suppose (by changing of f by some power of f) that chain recurrent set consists of the fixed points, that is $\mathcal{O}_i = p_i$ and $f|_{U_{p_i}}$ is conjugated with the diffeomorphism $a_{\lambda_{p_i}}(x_1, ..., x_\lambda, x_{\lambda+1}, ..., x_n) = (2x_1, 2x_2, ..., 2x_\lambda, 2^{-1}x_{\lambda_{p_i}+1}, 2^{-1}x_{\lambda_{p_i}+2}, ..., 2^{-1}x_n))$ by means a homeomorphism h_{p_i}.

A fixed point p of a flow (homeomorphism) f^t (f) is called *is topologically hyperbolic* if there exists a neighborhood $U_p \subset M^n$, a number $\lambda \in \{0, 1, ..., n\}$ and a homeomorphism $h_p : U_p \to \mathbf{R}^n$ such that $h_p f^t|_{U_p} = a^t_{\lambda_p} h_p|_{U_p}$ $(h_p f|_{U_p} = a_{\lambda_p} h_p|_{U_p})$ whenever the left and right sides are defined..

Below we prove each item of the theorem in a separate subsection.

All statements formulated for unstable manifolds hold for stable manifolds as well. One gets them if one formally changes “u” to “s” because $\mathcal{R}_f = \mathcal{R}_{f^{-1}}$ and stable manifolds of chain recurrent points for f are the unstable manifolds of the chain recurrent points for f^{-1}.

3.1. Representation of the ambient manifold as the union of the invariant manifolds of the periodic points

Proof of the item (1) of Theorem 1.

Proof. Now we prove that $M^n = \bigcup\limits_{i=1}^{k} W^u_{p_i}$ for every homeomorphism $f \in G$.

Let $x \in M^n$. Let us recall that a point $y \in M^n$ is called an *α-limit point* for the point x if there is a sequence $t_n \to -\infty,\ t_n \in \mathbb{Z}$ such that

$$\lim_{t_n \to -\infty} d(f^{t_n}(x), y) = 0.$$

The set $\alpha(x)$ of all α-limit points for the point x is called the *α-limit set* of x. As M^n is compact then the set $\alpha(x)$ is not empty. Let us show that $\alpha(x) \subset \mathcal{R}_f$. Indeed, as f is uniformly continuous and $\lim\limits_{t_n \to -\infty} d(f^{t_n}(x), y) = 0$, for every $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$ such that $d(f^{t_n}(x), y) < \varepsilon$ and $d(f^{t_n+1}(x), f(y)) < \varepsilon$ for every $n \geq n_\varepsilon$. Thus, $y, f(y), f^{t_n+1}(x), f^{t_n+2}(x), \dots, f^{t_{n+1}}(x), y$ is the ε-chain connected y with itself.

We show that $\alpha(x)$ consists of exactly one fixed point which depends on x. Assume the contrary i.e there are distinct fixed points $p_v,\ p_w \in \alpha(x)$. Since $\mathcal{R}_f$ is finite there is a $\rho > 0$ such that $d(p_i, p_j) > \rho$ whenever $i \neq j$. Denote $V_i = \{y \in M^n\ :\ d(y, p_i) < \frac{\rho}{3}\}$. Since all the points $p_i,\ i = \overline{1, k}$ are fixed there is a neighborhood U_i such that $cl(U_i) \subset V_i$ and $f^{-1}(cl(U_i)) \cap V_j = \emptyset$ for every $j \neq i$. By the assumption there is an increasing sequence q_ℓ of the iterations of f^{-1} such that $f^{-q_{2m}}(x) \in U_v$, $f^{-q_{2m+1}}(x) \in U_w$ and $q_{2m+1} - q_{2m} \geq 2$. We pick the sequence n_m so that n_m is the maximal natural number belonging to the interval (q_{2m}, q_{2m+1}) for each $f^{-(n_m-1)}(x) \in cl(U_v)$. Then

$f^{-n_m}(x) \notin cl(U_v)$. On the other hand $f^{-n_m}(x) = f^{-1}(f^{-(n_m-1)}(x)) \notin V_j$ for $j \neq v$ and hence $f^{-n_m}(x) \in (M^n \setminus \bigcup\limits_{i=1}^{k} U_i)$. But then $\alpha(x)$ is not a subset of $\mathcal{R}_f$ and we have a contradiction.

Thus for each point $x \in M^n$ there is the unique point $p_v(x) \in \mathcal{R}_f$ such that $\alpha(x) = p_v(x)$, i.e. there is a sequence $k_n \to +\infty$ such that $\lim\limits_{k_n \to +\infty} d(f^{-k_n}(x), p_v(x)) = 0$. It follows from the definition of the hyperbolic fixed point that $f^{-k_n}(x) \in W^u_{p_v(x)}$ for all n greater then some n_0. Then $x \in W^u_{p_v(x)}$ because the unstable manifold is invariant. $\square$

3.2. Embedding of the invariant manifolds of periodic points into the ambient manifold

To prove item (2) of Theorem 1 we need the following lemma.

Lemma 1. *Let σ be a hyperbolic saddle fixed point of a diffeomorphism $f \in G$, let $T_\sigma \subset W^s_\sigma$ be a compact neighborhood of the point σ and $\xi \in T_\sigma$. Then for every sequence of points $\{\xi_m\} \subset (M^n \setminus T_\sigma)$ converging to the point ξ there are a subsequence $\{\xi_{m_j}\}$, a sequence of natural numbers $k_{m_j} \to +\infty$ and a point $\eta \in (W^u_\sigma \setminus \sigma)$ such that the sequence of points $\{f^{k_{m_j}}(\xi_{m_j})\}$ converges to the point η.*

Proof. Without loss of generality one assumes $(U_\sigma \cap W^s_\sigma) \subset T_\sigma$, $\xi \in (U_\sigma \cap f(U_\sigma))$ and $\{\xi_m\} \subset (U_\sigma \cap f(U_\sigma))$. We pick a number $r > 0$ so that the ball $B_r(O) = \{(x_1, \dots, x_n) \in \mathbb{R}^n \ : \ (x_1^2 + \dots + x_n^2) \leq r\}$ would be a subset of the set $h_\sigma(U_\sigma)$.

Let $h_\sigma(\xi_m) = \bar{\xi}_m = (\bar{\xi}_{1,m}, \dots, \bar{\xi}_{\lambda_\sigma,m}, \bar{\xi}_{\lambda_\sigma+1,m}, \dots, \bar{\xi}_{n,m})$. The set $K^u = \{(x_1, \dots, x_{\lambda_\sigma}) \in Ox_1 \dots x_{\lambda_\sigma} \ : \ \frac{r^2}{4} \leq x_1^2 + \dots + x_{\lambda_\sigma}^2 \leq r^2\}$ is a fundamental domain of the restriction of the diffeomorphism a_{λ_σ} to $Ox_1 \dots, x_{\lambda_\sigma} \setminus O$. Then for every $m \in \mathbb{N}$ there is the unique integer k_m such that $\frac{r^2}{4} \leq 4^{k_m}\left((\bar{\xi}_{1,m})^2 + \dots + (\bar{\xi}_{\lambda_\sigma,m})^2\right) < r^2$. Let $\bar{\eta}_m = a^{k_m}_{\lambda_\sigma,+1}(\bar{\xi}_m)$. Since $\lim\limits_{m\to\infty} \bar{\xi}_m = h_\sigma(\xi) \in (Ox_{\lambda_\sigma+1} \dots x_n \setminus O)$ for every $i \in \{1, \dots, \lambda_\sigma\}$ the limit $\lim\limits_{m\to\infty} \bar{\xi}_{i,m}$ equals 0 and hence $\lim\limits_{m\to\infty} k_m = +\infty$. Furthermore the sequence $\{\bar{\xi}_{i,m}\}$ is bounded for every $i \in \{\lambda_\sigma + 1, \dots, n\}$ and hence $\bar{\eta}_{i,m} = \left(\frac{1}{2}\right)^{k_m} \bar{\xi}_{i,m} \to 0$ for $m \to +\infty$ and $i \in \{\lambda_\sigma + 1, \dots, n\}$.

Therefore the coordinates of the points $\bar{\eta}_m = (\bar{\eta}_{1,m}, \dots, \bar{\eta}_{n,m})$ satisfy $\frac{r^2}{4} \leq (\bar{\eta}_{1,m})^2 + \dots + (\bar{\eta}_{\lambda_\sigma,m})^2 < r^2$ for $i \in \{1, \dots, \lambda_\sigma\}$ and $\bar{\eta}_{i,m} \to 0$ as $m \to \infty$ for $i \in \{\lambda_\sigma + 1, \dots, n\}$, i.e. the points η_m are inside some compact subset of $\mathbb{R}^n$. Since any sequence of points of a compact set has a converging subsequence, there are a subsequence $\{k_{m_j}\}$ of the sequence $\{k_m\}$ and a point $\bar{\eta} \in (W^u_O \setminus O)$ such that $\lim\limits_{j\to\infty} \bar{\eta}_{m_j} = \bar{\eta}$. Then $\xi_{m_j} = h_\sigma^{-1}(a_{\lambda_\sigma}^{-k_{m_j}}(\bar{\eta}_{m_j}))$ is the desired subsequence. $\square$

Proof of item (2) of Theorem 1

Proof. Here we prove that $W^u_{p_i}$ is a submanifold of the manifold M^n, homeomorphic to $\mathbb{R}^{\lambda_{p_i}}$.

Let $T_{p_i} = h_{p_i}(E^u_{\lambda_{p_i}})$. Then for every point $x \in W^u_{p_i}$ there is a natural number n_x such that $x \in f^{-n_x}(T_{p_i})$. Let $T_{p_i}(x) = f^{-n_x}(T_{p_i})$ then there is a chart $\psi_x : U_x \to \mathbb{R}^n$ of the manifold M^n such that $\psi_x(U_x \cap T_{p_i}(x)) = \mathbb{R}^{\lambda_{p_i}}$. If $\lambda_{p_i} = n$ or $\lambda_{p_i} = 0$ then $\psi_x(U_x \cap T_{p_i}(x)) = \psi_x(U_x \cap W^u_{p_i})$. Therefore the unstable manifold of every node point is a smooth submanifold.

Now we show that $W^u_{p_i}$ is a submanifold of M^n homeomorphic to $\mathbb{R}^{\lambda_{p_i}}$ for every saddle point p_i as well. Suppose the contrary: $W^u_{p_i}$ is not a submanifold of M^n. Then there is a point $x \in W^u_{p_i}$ such that $(U_x \setminus T_{p_i}(x)) \cap W^u_{p_i} \neq \emptyset$ for every chart $\psi_x : U_x \to \mathbb{R}^n$ of the manifold M^n for which $\psi_x(U_x \cap T_{p_i}(x)) = \mathbb{R}^{\lambda_{p_i}}$. Hence there is a sequence $\{x_m\} \subset (W^u_{p_i} \setminus T_{p_i}(x))$ such that $d(x_m, x) \to 0$ for $m \to +\infty$.

Lemma 1 gives us that there is a subsequence x_{m_j} and there is a sequence k_j such that the sequence $y_j = f^{-k_j}(x_{m_j}) \subset W^u_{p_i}$ converges to a point $y \in (W^s_{p_i} \setminus p_i)$.

According to the item (1) of Theorem 1 there is a point $p_v \in \mathcal{R}_f$ such that $y \in W^u_{p_v}$. Consider three possibilities: [a] $\dim W^u_{p_v} = 0$; [b] $0 < \dim W^u_{p_v} < n$; [c] $\dim W^u_{p_v} = n$.

[a] If $\dim W^u_{p_v} = 0$ then $y_j \in W^u_{p_v}$ for all j starting from some one. Hence, $i = v$ and y is a homoclinic point, that contradicts to Statement 2. Thus case [a] is impossible.

[c] If $\dim W^u_{p_v} = n$ then $W^u_{p_v} = p_v$ and $y = p_v$, that contradicts to the condition $y \in W^s_{p_i}$. Thus case [c] is impossible.

[b] If $0 < \dim W^u_{p_v} < n$ then $v > i$ as f has no homoclinic points. According to Lemma 1 there is a subsequence $\{y_{j_r}\}$, a sequence $m_r \to +\infty$ and a point $z \in W^s_{p_v}$ such that the sequence $\{f^{m_r}(y_{j_r})\}$ converges to the point z. As $M^n = \bigcup\limits_{i=1}^{k} W^u_{p_i}$ then $z \in W^u_{p_w}$. Similarly to above arguments, $v \neq j, v \neq i$ and $0 < \dim W^u_{p_w} < n$. Due to finiteness of the set $\mathcal{R}_f$ the case [b] is also impossible.

Thus, $W^u_{p_i}$ is a topological submanifold of the manifold M^n homeomorphic to $\mathbb{R}^{\lambda_{p_i}}$. □

3.3. Asymptotic behaviour of the invariant manifolds of chain recurrent points

Proof of the item (3) of Theorem 1

Proof. Now we prove that $(cl(W^u_{p_i}) \setminus W^u_{p_i}) \subset \bigcup\limits_{j=1}^{i-1} W^u_{p_j}$.

If p_i is a sink then the set $cl(W^u_{p_i}) \setminus W^u_{p_i}$ is empty and the statement is automatically true. In the other cases let us consider a point $x \in (cl(W^u_{p_i}) \setminus W^u_{p_i})$ and prove that $x \in W^u_{p_v}$ for some $v < i$.

Indeed, as $x \in (cl(W^u_{p_i}) \setminus (W^u_{p_i} \cup p_i))$ then there is a sequence $\{x_m\} \subset W^u_{p_i}$ such that $d(x_m, x) \to 0$ for $m \to +\infty$. By item (1) of Theorem 1, $x \in W^u_{p_v}$ for some $v \in \{1, \ldots, k\}$. There are three possibilities: (a) p_v is a sink, (b) p_v is a saddle, (c) p_v is a source.

In the case (c) $x_m \in W^u_{p_v}$ for all m large enough. But then $p_v = p_i$ and $x \in W^u_{p_i}$ that contradicts the assumption.

In the case (a) $W^u_{p_v} = p_v$, $x = p_v$ and $x_m \in W^s_{p_v}$ for all m large enough. Then $W^u_{p_i} \cap W^s_{p_v} \neq \emptyset$ and $v < i$ is true.

In the case (b) by Lemma 1 there are a subsequence x_{m_j} and a sequence k_j such that the sequence $y_j = f^{-k_j}(x_{m_j})$ converges to a point $y \in (W^s_{p_v} \setminus p_v)$. By the item (1) of Theorem 1 there is a point $p_w \in \mathcal{R}_f$ such that $y \in W^u_{p_w}$, that is $p_v \prec p_w$. If $w = i$ then the statement is true. If not then arguing as above we get that the point p_w cannot be a source. The point p_w is evidently not a sink because p_v is a saddle point. Thus, the point p_w is a saddle different from p_v. Repeating the process, taking into account the finiteness of $\mathcal{R}_f$ and the absence of cycles, we obtain the statement in a finite number of steps. □

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