Динамические системы, 2018, том 8(36), №3, 263–274

MSC 2010: 37B05

How to check if one-dimensional solenoid in the sense of Williams can be realized as hyperbolic attractor of surface diffeomorphism

A. Yu. Zhirov
Moscow Aviation Institute (National Research University)
Moscow 125993. *E-mail: alexei_zhirov@mail.ru*

Abstract. In the paper, an algorithm is presented that allows for a given one-dimensional solenoid in the sense of Williams to answer the question: does a surface diffeomorphism with a one-dimensional hyperbolic attractor exist such that the restriction of the diffeomorphism to this attractor is topologically conjugate to the shift automorphism of the given solenoid.

Keywords: solenoid, hyperbolic attractor, branched manifold.

1. Introduction

In 1967 R. Williams [1] proposed the concept of generalized solenoid for modeling the dynamics generated by a diffeomorphism of a smooth manifold with a hyperbolic expanding attractor. The initial results of his theory are the following theorems. The conditions (S1–S4) specified in them will be formulated below.

Theorem 1 (Williams [2]). *Let M be a smooth manifold, $U \subseteq M$ be an open set, $f : U \to M$ be a smooth embedding, $\Lambda \subset U$ be a one-dimensional hyperbolic attractor for f. Then there exsist a branched 1-manifold K and a smooth map $\varphi : K \to K$ satisfying the conditions* (S1–S4) *such that the solenoid defined by the pair (K, φ) is homeomorphic Λ, and its shift automorphism is topologically conjugate to the restriction $f|_{\Lambda}$.*

Theorem 2 (Williams [2]). *Let K be a branched 1-manifold and $\varphi : K \to K$ be a smooth map satisfying the conditions* (S1–S4). *Then there exist a smooth manifold M, an open set $U \subseteq M$, a smooth embedding $f : U \to M$, and a one-dimensional hyperbolic attractor $\Lambda \subset U$ for f such that the solenoid defined by the pair (K, φ) is homeomorphic to Λ, and its shift automorphism is topologically conjugate to $f|_{\Lambda}$.*

For brevity, the statement of the theorem 2 will be said that the given *solenoid can be realizable as an attractor on the manifold M*.

Question. *Under what conditions additional to those of the theorem 2 it can be confirmed that* $\dim M = 2$*?*

In the proposed paper, an algorithm will be presented with which the following questions can be answered for given solenoid.

1. Whether the given solenoid is realizable as an attractor on an orientable surface and, if so, what is the minimal genus of a such surface? In this case the algorithm defines whether or not the embedding f generating the attractor preserves the orientation of the surface.

   If the answer to the first question is negative, then

2. Whether the given solenoid is realizable as an attractor on an non-orientable surface and, if so, what is the minimal genus of a such surface?

2. Definitions and combinatorial description of solenoid

Let K be finite graph (one-dimensional CW-complex) and a its edge (open 1-cell), *semi-edge* is component of the set $a \setminus \{x\}$, where $x \in a$. Two semi-edges are said to be equivalent if their intersection is semi-edge also. The equivalence class of semi-edges contained in a given edge is called its *end*. The end e of any edge is called to be *incident* to the vertex O if it is incident to semi-edge, which is a representative of the given end. The number of ends incident to a given vertex is called its *valence*.

Definition 1. Graph K is called *branched* (one-dimensional) *manifold* if the following conditions hold

(BM1) There are no vertexes of valency 1.

(BM2) For each vertex the partition of the set of ends of incident edges onto two nonempty disjunct subsets is fixed.

The ends belonging to one of these subsets will be called *outgoing* from a given vertex, and the ends belonging to the other one will be called *incoming* into it.

Vertices of the graph will be called *branch points* of branched manifold K.

The smoothness of branched manifold will be understood in the sense of the following definition of the smoothness of the arcs lying in it. By an arc in K we mean an image by the action of a continuous mapping γ of a segment of a real line into K, which is a local homeomorphism onto image.

Definition 2. The arc $\gamma \subset K$ is called to be *smooth* if for each t_0 such that $\gamma(t_0)$ is branch point there exists $\varepsilon > 0$ such that

1) arcs $\gamma|_{(t_0-\varepsilon,t_0)}$ and $\gamma|_{(t_0,t_0+\varepsilon)}$ are semi-edges belonging to ends of some edges incident to this vertex $\gamma(t_0)$;

2) these ends belong to distinct subsets onto which the set of all edges incident to $\gamma(t_0)$ is parted according to condition (BM2), i.e. one of this ends is incoming while the other is outcoming.

A continuous map of a branched manifold K into a branched manifold K' (into a smooth manifold M) is called *smooth* if the image of every smooth arc in K is a smooth arc in K' in the sense of previous definition (smooth arc in M in the usual sense).

Definition 3. Let K be branched manifold and $\varphi : K \to K$ be smooth map such that the following conditions hold

(S1) the set of all branch points of K is invariant under φ;

(S2) the non-wandering set of φ coincides with K;

(S3) φ is expanding in some metric;

(S4) for each branch point there exists a neighborhood U and an integer $m \geqslant 0$ such that $\varphi^m U$ is a smooth arc.

Solenoid in the sense of Williams is the limit of the inverse spectrum

$$K \xleftarrow{\varphi} K \xleftarrow{\varphi} K \xleftarrow{\varphi} \ldots,$$

and the homeomorphism $\varphi_\infty : K_\infty \to K_\infty$ defined by

$$\varphi_\infty \colon (x_1, x_2, x_3, \ldots) \mapsto (\varphi x_1, x_1, x_2 \ldots).$$

is called its *shift automorphism.*

Combinatorial description of the branched manifold 1) Let n be the number of edges of K. We enumerate in some way their ends by subscripts from the segment $\overline{1, 2n}$ of the natural series and define the map $\sigma : \overline{1, 2n} \to \overline{1, 2n}$, assuming that $j = \sigma(i)$ if i, j are the numbers of the ends of the same edge. It is clear that σ is a permutation on the set $\overline{1, 2n}$, decomposing into a product of cycles of period 2.

2) Let m be the number of vertices of the graph K. We enumerate them in some way by indices from the segment $\overline{1, m}$ of the natural series and denote by t_k^+ and t_k^-, respectively, the number of outgoing and the number of incoming ends of the edges for the vertex number k. Denote by $\mathcal{T} = (t_1^+, \ldots, t_m^+; t_1^-, \ldots, t_m^-)$ the sequence of $2m$ numbers thus defined.

Thus the branched manifold defines a pair $(\sigma, \mathbf{t})$. Conversely, each pair of $(\sigma, \mathbf{t})$, where σ is a permutation of the set $\overline{1, 2n}$, all cycles of which have a period 2, and $\mathcal{T} = (t_1^+, \ldots, t_m^+; t_1^-, \ldots, t_m^-)$ is a sequence of $2m$ natural numbers with the condition $\sum_{k=1}^m (t_k^+ + t_k^-) = 2n$ defines a branched 1-manifold. Therefore, we call the pair $(\sigma, \mathcal{T})$ associated with the branched manifold in the manner described above *configuration* of the branched 1-manifold. Of course, the configuration is ambiguous: it depends on the choice of the numbering of the ends of the edges, the numbering of the vertices, and which of the subsets of the ends in (BM2) are declared incoming and which are outgoing. Nevertheless, it is clear that knowing one of the configurations of a given branched manifold, one can write out all the others using a formal algorithm.

Combinatorial description of the map of branched manifold defining the solenoid. We enumerate the edges of the graph K with the index $i \in \overline{1, n}$, endow each of them with orientation, and denote the oriented edge number i by a_i. Through $\overline{a}_i$ we denote geometrically the same edge with opposite orientation. Let $\varphi : K \to K$ be a smooth map satisfying condition (S1). Then each edge a_i is partitioned by preimages of branch points into arcs such that the restriction of φ to each of them is one to one map onto some edge that can be overpassed in either a positive or negative direction. Therefore, with each edge a_i the word $w_i = \varphi(a_i)$ in the alphabet $\mathcal{A} = \{a_1, \dots, a_n; \overline{a}_1, \dots, \overline{a}_n\}$ can be associated. From the smoothness condition it follows that this word is irreducible, i.e. does not contain subwords of the form $a_i\overline{a}_i$ and $\overline{a}_i a_i$. Thus, with the map φ an ordered set of words $\mathbf{w} = (w_1, \dots w_n)$ be associated.

Conversely, a set of irreducible words $\mathbf{w} = (w_1, \dots w_n)$ defines a unique, up to homotopy, map φ of the graph K continuous on each edge and which is a local homeomorphism onto image in a neighborhood of each point that is not a branch point.

Now we establish the conditions under which this map is continuous at the branch points and satisfies the conditions (S1–S4) of the definition of solenoid.

Let's start with agreeing on the numbering of the vertices of the graph and its edges.

1. We number the vertices O_k with the index $k \in \overline{1, m}$.

2. Enumerate the ends of the edges, satisfying the condition: the outgoing ends for the vertex O_1 get numbers from 1 to t_1^+, outgoing ends for O_2 — from $t_1^+ + 1$ to $t_1^+ + t_2^+$ end so on; incoming ends for the vertex O_1 get numbers from $t + 1$ to $t + t_1^-$, incoming numbers ends to O_2 — from $t + t_1^- + 1$ to $t + t_1^- + t_2^-$ and so on, where $t = \sum_{k=1}^m t_k^+$.

3. We enumerate the edges a_i with the index $i \in \overline{1, n}$ lexicographically with respect to the numbering of their ends: the smaller is the minimal number of two ends of the edge, the smaller is the number of the edge.

4. Orient the edges considering the positive direction from the end with the smaller number to the end with the larger one.

The end of the edge a_i, which has a smaller number in the numbering of the ends (2), will be called *first* end, and the other end call the *second* one. These ends will be denoted by a_i and $\overline{a}_i$ respectively. (The fact that the same symbol denotes different things will not lead to misunderstandings, but it will be convenient.) The number of the first end of the edge a_i in the "continuous" numbering of the ends will be denoted $beta_-(i)$, and the number the second denote $\beta_+(i)$. The number of the edge that has one of its ends the end number j in continuous numbering will be denoted $\alpha(j)$. Thus, we obtain three maps of the sets of indices $\alpha : \overline{1, 2n} \to \overline{1, n}$; $\beta_-, \beta_+ : \overline{1, n} \to \overline{1, 2n}$. It is easy to see that these functions are determined by the conditions

$$\alpha(1) = 1, \quad , \alpha(i) = \alpha(\sigma(i)), \quad \alpha(i) < \alpha(j) \Leftrightarrow \min\{i, \sigma(i)\} < \min\{j, \sigma(j)\};$$

$$\beta_-(j) := \min\{i : \alpha(i) = j\}; \quad \beta_+(j) := \max\{i : \alpha(i) = j\};$$

The passage at the vertex O_k will be called an ordered pair of characters (two-letter word) in the alphabet $\mathcal{A} = \{a_i, \overline{a}_i : i \in \overline{1, n}\}$ denoting ends of edges in this vertex, such that one of them belongs to the upper partition and the other to the lower one.

We will call *pass* $\ell_i\ell_j$ *image* (where ℓ_i is a_i or $\overline{a}_i$, and ℓ_j is a_j or $\overline{a}_j$) to be two-letter word, the first letter of which is the last letter of the word $\varphi(\ell_i)$ and the second is the first letter of $\varphi(\ell_j)$.

Then the condition of continuity of the map φ be

(SC0) Each two-letter subword of each word $w_i = \varphi(a_i)$ is a passage at some vertex.

And the condition of smoothness of the map φ be

(SC1) For each vertex, all its passes images are the passages at the same vertex.

Thus, calculating the images of passes, we can check the continuity condition φ, and at the same time to find the map of the vertices ϖ defined in the first part of the following

Definition 4. 1. *The map of the vertices* ϖ is such permutation that

$$\varphi(O_k) = O_{\varpi(k)}.$$

2. *The intersection matrix* $G = (g_{ij})$ of the map φ is defined by

$$g_{ij} := \#\{a_i, \overline{a}_i \in w_j\},$$

where # denotes the number of elements in finite set.
3. *The intersection vector* $\mathbf{g} = (g_i)$ of the map φ is defined by

$$g_i := \sum_j g_{ij}.$$

4. The sequence $\mathcal{E} = \{\varepsilon_1, \ldots, \varepsilon_m\}$ of signs $\varepsilon_k = \pm$, depending of the way by which the map φ turns the directions of passages on the vertex O_k into directions of passages in $O_{\varpi(k)}$:

$$\varepsilon_k = \begin{cases} +, & if \text{ “ingoing} \to \text{outgoing”} \;\mapsto\; \text{“ingoing} \to \text{outgoin”;} \\ -, & if \text{ “ingoing} \to \text{outgoin”} \;\mapsto\; \text{“outgoing} \to \text{ingoing”.} \end{cases}$$

Lemma.
1. *Conditions* (S2) *and* (S3) *together are equivalent to*
(SC2) *The intersection matrix G is primitive i.e. some power of it is a positive matrix.*
2. *Condition* (S4) *is equivalent to*
(SC3) *for each vertex, some iteration of all its passes is the same pass,* up to the reverse.

Proof. According to the Perron-Frobenius theorem primitiveness and integerness of the matrix G imply that it has eigenvalue which is greater than one (Perron eigenvalue), which corresponds to positive eigenvector (Perron eigenvector). Therefore, it is possible to define on K a Lebesgue measure such that the length of each edge be equal to the corresponding component of the left Perron eigenvector, and the map will be a uniform expanding with a coefficient equal to Perron eigenvalue. Since φ is expanding and the matrix G is primitive, it does not have open invariant sets and the periodic points of φ are dense in K. So primitiveness of the matrix implies conditions (S2,S3). The converse, as well as the second statement of the lemma, is obvious. □

All the above means that each one-dimensional solenoid define the set of data $\langle\sigma, \mathcal{T}; \mathbf{w}\rangle$ by which it is possible to find the matrix G and the permutation ϖ. This makes the following definition natural.

Definition 5. The collection of data $\langle\sigma, \mathcal{T}; \mathbf{w}\rangle$, where σ is permutation on $\overline{1,n}$ with all cycles of the length 2; $\mathcal{T} = (t_1^+, \dots, t_m^+; t_1^-, \dots, t_m^-)$ is the sequence of $2m$ positive integers ($m \geqslant 1$) with the sum equal to $2n$; $\mathbf{w}$ is the ordered set of irreducible words in alphabet $\mathcal{A} = \{a_i, \overline{a}_i : i \in \overline{1,n}\}$ is called *solenoidal code* if the condition (SC0–SC3) holds.

Of course, the code is determined by the solenoid ambiguously because of the following elements of arbitrariness

1. The choice of the numbering of vertexes.
2. The choice of which of the two possible subsets of the set of the ends of the edges incident to each vertex is considered to be the outgoing (ingoing).
3. The numbering of each of the sets of outgoing and ingoing edges of the edges.

In what follows, considering the solenoid given by any of its codes, we will save the first two choices while we will have to consider all the codes obtaining while changing the choice of edge ends.

This means that the sequence will not change while the σ permutation changes. The latter implies a change in the sequence of words $\mathbf{w}$ occurring due to a corresponding change in the numbering of the edges of the graph and a possible change in their orientations. The change in the numbering of the ends of the edges is determined by the permutation $\tau : \overline{1,2n} \hookleftarrow$, which leaves invariant each of the subsets of the set $\overline{1,2n}$ of indices of the form

$$\overline{t_{k-1}^+ + 1, t_k^+} \quad (k \in \overline{1,m},\ t_0^+ = 0)$$

and

$$\overline{t + t_{k-1}^- + 1, t + t_k^-} \quad (k \in \overline{1,m},\ t_0^- = 0,\ t = \sum_{k=1}^m t_k^+).$$

Definition 6. We will call the permutation τ of $\overline{1,2n}$ which leaves invariant the above subsets to be *admissible*.

If the admissible permutation is given we can to calculate the changed code $\langle\sigma^\tau, \mathcal{T}; \mathbf{w}^\tau\rangle$ ($\mathcal{T}$ is the same) as follows.

The permutation of σ is transformed into a permutation of σ^τ, the cycles of which are obtained from the cycles of σ follows. Let (i, j) be a cycle σ. Then it corresponds to the cycle $(\tau(i), \tau(j))$ of σ^τ. Of course it means that $\sigma^\tau = \tau^{-1} \circ \sigma \circ \tau$.

To calculate the sequence of words $\mathbf{w}^\tau$, we note that the letters of the words w_j correspond to the cycles σ. Namely, if $(i, \sigma(i)$, where $i < \sigma(i)$, then this cycle determines some letter a_j. We will write $a_j \simeq (i, \sigma(i))$, and the index of this letter can be defined as follows. We write the permutation σ in the form of a product of such cycles, order the factors by the first index and enumerate them according to this ordering. Then j is the number of the cycle $(i, \sigma(i))$ in this numbering. Keeping this ordering of cycles, replace each of them with the cycle $(\tau(i), \tau(\sigma(i)))$, keeping a sequence of characters in each cycle. The words w_k^τ which define the map φ $a'_k \mapsto w_k^\tau$ must be written in the alphabet $\mathcal{A}' = \{a'_i, \overline{a'}_i : i \in \overline{1,n}\}$ the letters of which denote the same edges of the graph, renumbered and, possibly, with a changed orientation. The letter $a'_{j'}$, denoting the same edge of the graph as the corresponding letter a_j, is defined by the cycle $(\tau(i), \tau(\sigma(i)))$ and its number j' is the number of this cycle in their numbering in ascending smaller the indices of the permutation cycles of σ^τ. The orientation of this edge coincides with its initial orientation i.e. $a'_{j'} = a_j$, if $\tau(i) < \tau(\sigma(i))$, and opposite to it otherwise, i.e. $a'_{j'} = a_j^{-1}$.

Thus, to find the words $w_{j'}^\tau$ that define the map of φ in new numbering of the ends is necessary in the correspondences $a_j \mapsto w_j$ to make the appropriate replacement of letters a_j and a_j^{-1} on $a'_{j'}$ and $a_{j'}^{-1}$.

Definition 7. We will say that the solenoid code obtained from given code by means of the described procedure using the admissible permutation τ is *τ-equivalent* to the initial one.

3. Band surfaces and codes of attractors

The solenoidal representation of a expanding one-dimensional hyperbolic attractor Λ of a diffeomorphism $f : U \to M$ in Williams Theorem 1 is obtained as follows. A tubular neighborhood N of the set Λ which is fibered onto $(\dim M - 1)$-disks lying in the stable manifolds of the points of Λ is constructed. The partition of N onto these disks as leaves is semi-invariant with respect to f so that corresponding factorization of N defines branched manifold and its expanding map which define the solenoid.

In the case $\dim M = 2$ the tubular neighborhood may be constructed in the form of *band surface* [3] i.e the surface Π with boundary arranged as follows.

1. There is a finite collection of simple disjoint arcs (called base segments of Π) $I_1, \dots, I_m$ of stable manifolds of some periodic points of f each of them containing corresponding periodic point. This collection is semi-invariant i.e. $f(I_k) \subseteq I_{\varpi(k)}$, where ϖ is some permutation on $\overline{1,m}$.

2. There are two partitions $\mathcal{D}_k^+$ (upper partition) and $\mathcal{D}_k^-$ (lower partition) of each base segment I_k onto t_k^+ and t_k^- segments ($t_k^+, t_k^- \geqslant 1$). Along some pairs $\Delta, \Delta' \in \mathcal{D} := \bigcup_k \mathcal{D}_k^+ \cup \mathcal{D}_k^-$ of these segments a *band* $\Pi_{\Delta,\Delta'}$ is glued to the base $I := \bigcup_k I_k$. The band surface is union of this bands. What is more, if $x \in \Lambda \cap \Pi_{\Delta,\Delta'}$ then the connected component of unstable manifold intersection $W^u(x) \cap \Pi_{\Delta,\Delta'}$ containing x is the arc with endpoints on segments Δ, Δ' (called *ends* of the band) and the connected component

of stable manifold intersection $W^s(x) \cap \Pi_{\Delta,\Delta'}$ containing x is the arc with endpoints on two other opposite edges of $\Pi_{\Delta,\Delta'}$ (called *margins* of band). The collection of all such arcs of stable manifolds defines partition of Π onto 1-discs which is semi-invariant and define via factorization the branched 1-manifold K and its map φ satisfying to the conditions of definition of a solenoid.

We will also need a combinatorial description of the map $f|_\Pi$ introduced in [3] to be a collection of data $\langle \sigma, \mathcal{T}, \mathbf{o}; \varpi, \mathcal{E}^s, \mathcal{E}^u, \mathbf{g}, \mathcal{L} \rangle$ called *the code of the attractor* Λ with respect to band surface Π. These data formally are

1. σ is permutation on $\overline{1,n}$ (n the number of bands) all whose cycles are of period 2;
2. $\mathcal{T} = (t_1^+, \ldots, t_m^+; t_1^-, \ldots, t_m^-)$ is a sequence of $2m$ natural numbers (m is the number of base segments);
3. $\mathbf{o} = \{o_1, \ldots, o_n\}$ — sequence of signs $\pm$;
4. ϖ — permutations on $\overline{1,m}$;
5. $\mathcal{E}^s = \{\varepsilon_1^s, \ldots, \varepsilon_n^s\}$, $\mathcal{E}^u = \{\varepsilon_1^u, \ldots, \varepsilon_n^u\}$ — sequences of signs $\pm$;
6. $\mathbf{g}$ — integer n-vector of natural entries;
7. $\mathcal{L} = \{l_1, \ldots, l_n\}$ finite integer sequence.

Definition 8. The data set listed above is called a *formal code.*

To define the geometric meaning of these data let us we introduce the following agreements.

Let us fix a positive direction (say "from the left to right") on each base segment; positive transverse direction (say "from bottom to top") to each of them. In the case of orientable surface suppose that each pair of these direction define the same orientation of the surface.

Now we assign the numbers to the ends $\Delta \in \mathcal{D}$ of the bands so that for $\Delta_i \in \mathcal{D}_j^{\varkappa}$ and $\Delta_{i'} \in \mathcal{D}_{j'}^{\varkappa'}$ ($\varkappa, \varkappa' = \pm$) $i < i'$ if and only if either $\varkappa = +, \varkappa' = -$ or $\varkappa = \varkappa'$ and $j < j'$ or $\varkappa = \varkappa'$, $j = j'$ and Δ_i lies on I_j to the left of $\Delta_{i'}$.

So the permutation σ be well defined and it means that the bands are glued to the base segments along the pairs of ends of the form $(\Delta_i, \Delta_{\sigma(i)})$. The bands are numbered in such a way that the band Π_j is glued to segments $\Delta_{\beta_-(j)}, \Delta_{\beta_+(j)}$ ($\beta_-(j) < \beta_+(j)$, $\alpha(\beta_-(j)) = \alpha(\beta_+(j)) = j$).

The numbers t_k^+, t_k^- of the sequence $\mathcal{T}$ denote the numbers of ends of bands which are glued to the base segment I_k to the top and bottom, respectively.

The signs $o_i = -$ of the sequence $\mathbf{o}$ means that the band Π_i is twisted when gluing and $o_i = +$ in other case. In the case of an orientable surface, there are no twisted tapes, while for non-orientable ones they should be.

The triple $(\sigma, \mathcal{T}, \mathbf{o})$ is called *the configuration* of the band surface.

The remaining data of the code characterize the action of the map $f_{\prod}$ on the base segments and bands.

Permutation ϖ shows that $f_{\prod} I_k = I_{\varpi(k)}$.

Signs ε_k^s are "+"ore "−"depending of whether $f_{\prod}$ move positive direction of I_k into positive ore negative direction of $I_{\varpi(k)}$. Signs ε_k^u means the same with respect transverse directions.

Component g_i of the vector $\mathbf{g}$ is the common number of component of intersection of the band Π_i with images of other bands.

Element l_k of the sequence $\mathcal{L}$ is she number of components of intersection of images of all bands with that component of $I_k \setminus f(I_{\varpi^{(-1)}})$, which is to the left of $f(I_{\varpi^{-1}})$ ($l_k = 0$ if left end of $f(I_{\varpi^{-1}})$ coincides with that one of I_k).

In [3], it is shown (Lemma 2.4.1) that the code of the attractor satisfies some conditions that need to be formulated here. This requires the following notation. By formal configuration $(\sigma, \mathcal{T}, \mathbf{o})$ define integral vectors $\mathbf{V}_+^j$ ($j \in \overline{0,t}$) and $\mathbf{V}_-^j$ ($j \in \overline{t, t+u}$, $t = \sum t_j^+$, $u = \sum t_j^-$)

$$\mathbf{V}_+^0 = \mathbf{V}_-^0 := \mathbf{0}; \quad \mathbf{V}_\pm^j := \mathbf{V}_\pm^{j-1} + \mathbf{e}^{\alpha(j)}.$$

Here $\mathbf{e}^i$ is the vector with all entries 0 except for i-th which is 1.

$$\mathbf{v}_+^k := \mathbf{V}_+^{T_k} - \mathbf{V}_+^{T_{k-1}}; \quad \mathbf{v}_-^k := \mathbf{V}_+^{T_{m+k}} - \mathbf{V}_+^{T_{m+k-1}}, \quad k \in \overline{1,m}$$

Here $T_k = \sum_{j=1}^k t_j^+$ for $k \leqslant m$ and $T_{m+k} = t + \sum_{j=1}^k t_j^-$.

In these terms, the conditions mentioned are

(FC1) $\langle \mathbf{V}_+^{\varpi(k)} + \mathbf{V}_-^{\varpi(k)}, \mathbf{g} \rangle \geqslant t_k^+ + t_k^-$;

(FC2) $\langle \mathbf{V}_+^{\varpi(k)} - \mathbf{V}_-^{\varpi(k)}, \mathbf{g} \rangle = \varepsilon_k^u (t_k^+ - t_k^-)$;

(FC3) $0 \leqslant l_{\varpi(k)} \leqslant \frac{1}{2}\Big(\langle \mathbf{V}_+^{\varpi(k)} + \mathbf{V}_-^{\varpi(k)}, \mathbf{g} \rangle - (t_k^+ + t_k^-)\Big)$.

Here $\langle \cdot, \cdot \rangle$ denotes the usual scalar product.

Definition 9. The formal code is said to be *admissible* if it satisfies to conditions (FC1–FC3).

In [3] the conditions (FC1–FC2) immediately included in the definition of a formal code.

We note that condition (FC3) is especially important now, since it implies the finiteness of the set of formal codes with given n and m, and an estimate of their number can be easily obtained.

Solenoidal representation of attractor Λ is obtained from band surface Π and the map $f|_{\prod}$ as follows. Branch points (vertexes) of branch manifold K are images under factorization of base intervals I_k and them are numbered by the same indexes. So, the permutation ϖ is the same as permutation of branch points. Partitions of ends edges of K onto ingoing and outgoing ones corresponds to partitions of segments I_k onto upper and lower ends of bands $\mathcal{D}_k^+$ and $\mathcal{D}_k^-$. Edges a_i of K are factors of bands Π_i

and are enumerated y the same indexes. Orientation of the edge a_i corresponds to the "vertical"direction on Π_i i.e. direction from its end $\Delta_{\beta_-(i)}$ to $\Delta_{\beta_+(i)}$.

The Theorem 3.1.1 [3] implies that each solenoidal representation of an attractor is obtained by the method described above.

4. Algorithm for checking the realizability of a solenoid on the surface

The data $\sigma, \mathcal{T}, \mathbf{w}$ which give combinatorial description of solenoid enables to find a part of the data composing the code of the attractor: $\sigma, \mathcal{T}, \varpi, \mathbf{g}, \mathcal{E}^u = \mathcal{E}$. Hence, if the solenoid is given, it is possible to enlarge these data to obtain the formal code. This can be done in various ways of specifying missing data $\mathbf{o}, \mathcal{E}^s, \mathcal{L}$. What is more, in the source data we can replace σ and $\mathbf{w}$ by performing a permissible permutation τ of the indices. In any case, the number of opportunities to do all this is finite and can be estimated through initial data. Thus, it is possible to consider all the various additions of the solenoidal code to the formal code.

In [3] an algorithm for checking if the formal code is realizable by some attractor via some band representation of it is described. It is based on the some ancillary algorithms (subroutines in terms of computer sciences) i.e. so called Algorithms A, B and D. We will not give a description of these algorithms here because it requires too much space, but it is necessary to specify their input and output data along with a brief comments.

Algorithm A.

Input. Formal (n, m)-configuration $(\sigma, \mathcal{T}, \mathbf{o})$.

Output. 1. The sequence $\mathcal{B} = \{b_0, b_1, \ldots\}$ of nonnegative integers such that almost all (but not all) of them are equal to zero.

2. The finite set of nonempty words $\mathcal{R} = \{\rho^1, \ldots, \rho^b\}$ in alphabet $\mathcal{A} = \{a_i, \overline{a}_i : i \in \overline{1, n}\}$.

Comment. In the case when the configuration on input is indeed that one of some band surface of the attractor we have $b_0 = 0$ (it is essential for our goals), while other elements of $\mathcal{B}$ define so called boundary type of the attractor (see [3] for definition, we need not for it now). The set of words $\mathcal{R}$ defines some co-representation $\langle a_1, \ldots, a_n | \rho^1, \ldots, \rho^b \rangle$ of fundamental group of the band surface.

Algorithm B.

Input. Formal (n, m)-code $\langle \sigma, \mathcal{T}, \mathbf{o}; \varpi, \mathcal{E}^s, \mathcal{E}^u, \mathbf{g}, \mathcal{L} \rangle$

Output.

1. Formal configuration $(\widehat{\sigma}, \widehat{\mathcal{T}}, \widehat{\mathbf{o}})$.
2. Ordered set $\widehat{\mathbf{w}} = \{\widehat{w}_1, \ldots, \widehat{w}_n\}$ in alphabet $\mathcal{A} = \{a_i, \overline{a}_i : i \in \overline{1, n}\}$.
3. Sequence $\widehat{\mathcal{L}} = \{\widehat{l}_1, \ldots, \widehat{l}_m\}$ of non-negative integers.

Comment. If the formal code on input is indeed a code of some attractor then the data on output $(\widehat{\sigma}, \widehat{\mathcal{T}}, \widehat{\mathbf{o}})$ and $\widehat{\mathcal{L}}$ are the same as on input, while the sequence of words $\widehat{\mathbf{w}}$ is that one which define the map $\varphi : K \to K$ which defines solenoid. In any case $\widehat{\mathbf{w}}$

defines in the same way as $\mathbf{w}$ some integer matrix $\widehat{G}$ and the corresponding vector $\widehat{\mathbf{g}}$. We will need of them to check the condition of realizability.

Algorithm D.
Input. Formal (n,m)-code $\langle\sigma,\mathcal{T},\mathbf{o};\varpi,\mathcal{E}^s,\mathcal{E}^u,\mathbf{g},\mathcal{L}\rangle$; integer $N>1$.
Output. Formal (n,m)-code $\langle\sigma,\mathcal{T},\mathbf{o};\varpi^{(N)},\mathcal{E}^{s(N)},\mathcal{E}^{u(N)},\mathbf{g}^{(N)},\mathcal{L}^{(N)}\rangle$

Comment. If the formal code on input is indeed a code of some attractor of f then the code on output is the code of the same attractor considered as that one with respect to N-th iteration of f.

This algorithm of checking realizability of solenoid as an attractor on surface is based on the following criteria.

Theorem 3 ([3], Theorems 3.8.1, 3.8.2)**.** *The following conditions are necessary and sufficient for the realizability of the formal code* $\langle\sigma,\mathbf{t},\mathbf{o};\varpi,\mathcal{E}^s,\mathcal{E}^u,\mathbf{g},\mathcal{L}\rangle$.

(RC0) *The number b_0 calculated by algorithm A is zero and the formal configuration $(\widehat{\sigma},\widehat{\mathcal{T}},\widehat{\mathbf{o}})$ calculated by algorithm B coincides with the initial one.*

(RC0) *The vector composed of the sum of rows of the matrix $\widehat{G}$ calculated by Algorithm B coincides with* $\mathbf{g}$.

(RC2) *None of the words in the set $\{\widehat{\varphi}(\rho):\rho\in\mathcal{R}\}$ is cyclically trivial. Here $\widehat{\varphi}$ is the map $a_i\mapsto\widehat{w}_i$.*

(RC3) *The matrix $\widehat{G}$ is primitive.*

(RC4) *For minimal $N\in\mathbb{N}$ such that $\varpi^{(N)}=\mathrm{id}$, $\mathcal{E}^{s(N)}=\mathcal{E}^{u(N)}=+$ and for each $k\in\overline{1,m}$ the following conditions hold*

$$\begin{aligned}(\mathbf{V}_+^{T_{k-1}+i}-\mathbf{V}_+^{T_{k-1}})\mathbf{g}^{(N)}&\neq l_k^{(N)}+i\qquad \textit{for all } i\in\overline{1,t_k^+-1}\\(\mathbf{V}_-^{T_{m+k-1}+i}-\mathbf{V}_-^{T_{m+k-1}})\mathbf{g}^{(N)}&\neq l_k^{(N)}+i\qquad \textit{for all } i\in\overline{1,t_k^--1}\end{aligned}$$

Now we are ready to formulate the desired algorithm.

Main algorithm

Input. An abstract 1-solenoid defined by a branched manifold with m branch points and n edges given by its solenoidal code $\langle\sigma,\mathcal{T};\mathbf{w}\rangle$. (The code must satisfy the conditions (SC0–SC3)).

<u>Move 0.</u> By the sequence of words $\mathbf{w}$ define the permutation ϖ and the sequence of signs $\mathcal{E}$ (Definition 4).

<u>Move 1.</u> Enumerate all admissible permutations of τ and for each (including the identity) calculate the transformed code $\langle\sigma^\tau,\mathcal{T};\mathbf{w}^\tau\rangle$.

<u>Move 2.</u> For each of them (we omit the index τ below) define a vector $\mathbf{g}$ by a sequence of words $\mathbf{w}$ and write down all formal codes $\langle\sigma,\mathbf{t},\mathbf{o};\varpi,\mathcal{E}^s,\mathcal{E}^u,\mathbf{g},\mathcal{L}\rangle$ with $\mathbf{o}=(+,\dots,+)$, $\mathcal{E}^u=\mathcal{E}$, $\mathcal{E}^s=\mathcal{E}^u$ or $\mathcal{E}^s=-\mathcal{E}^u$ and $\mathcal{L}$ satisfying to the condition (FC3).

Move 3. For each of these formal codes check the conditions of Theorem C and, if they are not satisfied for any of them, to conclude that this solenoid is *not realizable on any oriented surface*. Go to move 6.

Move 4. For each code that satisfies the conditions of Theorem C, check equality $\widehat{\mathbf{w}} = \mathbf{w}$, where $\widehat{\mathbf{w}}$ is the sequence of words which is calculated for using Algorithm B and $\mathbf{w}$ is that one from initial solenoidal code. If it is not satisfied for any of them, make the same conclusion.

Move 5. If we find a code that satisfies all these conditions then the solenoid is *realizable on orientable surface*. And calculating for these codes (using the Algorithm A) a set of words $\mathcal{R}$ we define via corresponding co-representation of fundamental group topological types of these surfaces. So we obtain minimal genus of such surface. What is more, if $\mathcal{E}^s = \mathcal{E}^u$ then generating the attractor diffeomorphism preserves orientation and it reverses it in the case $\mathcal{E}^s = -\mathcal{E}^u$. *The end of the procedure of the Algorithm.*

Move 6. For each of transformed solenoidal codes calculated on the Move 1 write down all formal codes with all sequences $\mathbf{o}$ with at least one entry "$-$", $\mathcal{E}^u = \mathcal{E}$, all possible sequences $\mathcal{E}^s$ and all sequences $\mathcal{L}$ satisfying to the condition (FC3).

Moves 7 and 8. For each of these codes check the same conditions as on the Moves 4 and 5. And if these are not valid for each of them than the solenoid is *not realizable on any surface*.

Move 9. If we find a code that satisfies all these conditions then the solenoid is *realizable on non-orientable surface* and we can define minimal genus of such one in the same way as in orientable case using Algorithm A.
The end of the procedure of the Algorithm.

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Получена 05.10.2018