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On construction of axiom A 3-diffeomorphism with one-dimensional surface attractor-repeller dynamics[1]

M. Barinova, V. Grines, O. Pochinka

Higher School of Economics

Nizhny Novgorod 603005. *E-mail: mkbarinova@yandex.ru, vgrines@yandex.ru, olga-pochinka@yandex.ru*

Abstracts. We suggest a method of a construction of axiom A 3-dieomorphisms whose non-wandering set consists of exactly one-dimensional surface attractor and one-dimensional surface repeller. Unlike from examples constructed by Ch. Bonatti, N. Guilman and Sh. Yi, our diffeomorphisms are not structurally stable, however suggested method gives rather simple construction of new types of 3-manifolds, admitting "hyperbolic sink-hyperbolic source" dynamics.

Keywords: A-diffeomorphism, surface basic set

1. Introduction

Let M be a closed n-manifold and $f : M \to M$ be an axiom A diffeomorphism. By Smale's spectral theorem the non-wandering set of f consists of finite number f-invariant closed subsets, named *basic sets*. For a basic set Λ a pair (a, b), where $a = \dim W_\Lambda^u, b = \dim W_\Lambda^s$ is called *type of the basic set*.

A basic set A of the diffeomorphism f is called *attractor* if it has a *trapping region*, that is a compact neighborhood $U_A \subset M$ such that $f(U_A) \subset int\, U_A$ and $\bigcap_{j \in \mathbb{N}} f^j(U_A) = A$. A basic set R is called a *repeller* if it is an attractor for f^{-1}.

A hyperbolic attractor of diffeomorphism f is called *surface* if there exists a compact surface Σ that $A \subset \Sigma$, $f(\Sigma) \subset \Sigma$. A surface repeller is defined as surface attractor for f^{-1}.

There is a natural question: does some manifold admit an A-diffeomorphism with exactly two basic sets of the same dimension and attractor-repeller dynamics? Such a diffeomorphism is automatically Ω-stable. The simplest example of such a system is a Morse-Smale diffeomorphism on n-sphere ($n \geq 1$) whose non-wandering set consists of exactly one sink and one source. These examples are structurally stable and exhaust all possible diffeomorphisms with zero-dimensional attractor and repeller.

For an attractor and repeller with the dimension $2k$ ($k \in \mathbb{N}$) also not so difficult to realize a diffeomorphism on n-manifold ($n > 2k$) as a direct product of Anosov diffeomorphism on $2k$-torus with the type (k, k) by sink-source diffeomorphism on

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$(n-2k)$-sphere. Moreover, for 3-manifolds by V. Grines, Yu. Levchenko, V. Medvedev, O. Pochinka [4] proved that two-dimensional attractor-repeller 3-diffeomorphisms there are only on mapping torus and obtained complete topological classification of such rough systems.

Particular case is a surfaces diffeomorphism with one-dimensional attractor and repeller. Such dynamics is achieved, for example, by taking of the connected sum of two DA-models on 2-tori. However R. Robinson, R. Williams [5] proved that among such diffeomorphisms there are no structural stable one.

B. Jiang, Y. Ni and S. Wang [2] proved that a 3-manifold M admits an axiom A diffeomorphism f whose non-wandering set consists of solenoid attractors and repellers if and only if M is a lens space $L(p,q)$ with $p \neq 0$. They also shown that such f are not structural stable. C. Wang and Y. Zhang [6] got infinitely many genus two 3-manifolds, each admits a diffeomorphism whose non-wandering set consists of two Williams solenoids, one attractor and one repeller. These manifolds contain half of Prism manifolds, Poincare's homology 3-sphere and many other Seifert manifolds, all integer Dehn surgeries on the figure eight knot, also many connected sums.

On the other hand, due to Ch. Bonatti, N. Guelman [1], Shi Yi [3], there are examples of rough 3-diffeomorphisms with one-dimensional attracor and repeller. But all examples have very complicated descriptions.

In this paper we suggest a method of a construction of an axiom A 3-diffeomorphisms whose non-wandering set consists of exactly one-dimensional surface attractor and one-dimensional surface repeller. All known examples were constructed in [1] and [3]. Constructed in this paper diffeomorphisms are not structurally stable, however suggested method gives rather simple construction of new 3-manifolds, admitting "hyperbolic sink-hyperbolic source" dynamics, different from the manifolds constructed in [1] and [3].

2. Construction

The diffeomorphism will be constructed step by step in this section as following:

- take an Anosov diffeomorphism of a 2-torus $\mathbb{T}^2$;

- make a Smale "surgery operation" to obtain the system with one fixed source and one-dimensional attractor on the torus;

- multiply $\mathbb{T}^2$ by $\mathbb{R}$ with contraction to 0, hence one-dimensional surface attractor on $\mathbb{T}^2 \times \mathbb{R}$ will be obtained;

- construct a fundamental domain of the attractor;

- take an analogical sample with one-dimensional surface repeller;

- "glue" the fundamental domains of the diffeomorphisms in the basins of attractor and repeller in accordance with dynamics.

2.1. Anosov diffeomorphism of a 2-torus

Let $C \in GL(2, \mathbb{Z})$ be a hyperbolic matrix with the eigenvalues λ_1, λ_2 so that $\lambda = |\lambda_1| > 1$ and $|\lambda_2| = 1/\lambda$. As the matrix C has the determinant equals 1 then it generates the hyperbolic automorphism $\widehat{C} : \mathbb{T}^2 \to \mathbb{T}^2$ with the fixed point O. This automorphism is Anosov diffeomorphism, so there are two transversal foliations (stable and unstable) which are dense on the 2-torus, and a set of periodical points is also dense.

2.2. Smale "surgery operation"

Let $U(O) \subset \mathbb{T}^2$ be a some neighbourhood of the fixed point O of the diffeomorphism $\widehat{C}$ and x, y be local coordinates such that the diffeomorphism $\widehat{C}$ in these coordinates has a form

$$\widehat{C}(x, y) = (x/\lambda, \lambda y).$$

Then $Ox \subset W_O^s$ and $Oy \subset W_O^u$, also $\{y = const\}$ and $\{x = const\}$ are stable and unstable foliations. A diffeomorphism $\widehat{B} : \mathbb{T}^2 \to \mathbb{T}^2$ with properties described below will be constructed in this section:

- $\widehat{B}$ will be identity on $\mathbb{T}^2 \setminus U(O)$;

- $\widehat{B}$ will keep unstable manifolds of $\widehat{C}$ everywhere;

- $\widehat{B}$ will add additional expansion along stable manifolds of $\widehat{C}$ inside some neighbourhood of O;

- the composition $\widehat{\Psi} = \widehat{B} \circ \widehat{C}$ is DA-diffeomorphism with a fixed source O and one-dimensional attractor A.

Let $\nu : [0, 1] \to [0, 1]$ be a diffeomorphism defined by the graph on Fig. 1. Then $\nu_t(x) = (1 - t)x + t\nu(x)$, $t \in [0, 1]$ is an isotopy between the identity map $\nu_0(x)$ on $[0, 1]$ and $\nu_1(x) = \nu(x)$. Let $\sigma(x, a, b) : \mathbb{R} \to \mathbb{R}$ be a sigmoid function of the form

$$\sigma(x, a, b) = \begin{cases} \frac{1}{1 + \exp\left(\frac{(a+b)/2 - x}{(x-a)^2(x-b)^2}\right)}, & a < x < b, \\ 0, & otherwise. \end{cases}$$

It monotonically sends $[a, b]$ to $[0, 1]$.

For every $t \in [0, 1]$ let us define a diffeomorphism $B_t : [0, 1]^2 \to [0, 1]^2$ which possess the symmetries with respect to both axis Ox, Oy and in the first quadrant given by the formula

$$B_t(x, y) = \begin{cases} (\nu_t(x), y), & 0 \leqslant y < \frac{1}{2\lambda^3}, \\ \left(\sigma(y, \frac{1}{2\lambda^3}, \frac{1}{2})x + (1 - \sigma(y, \frac{1}{2\lambda^3}, \frac{1}{2}))\nu_t(x), y\right), & \frac{1}{2\lambda^3} \leqslant y < \frac{1}{2}, \\ (x, y), & \frac{1}{2} \leqslant y \leqslant 1. \end{cases}$$
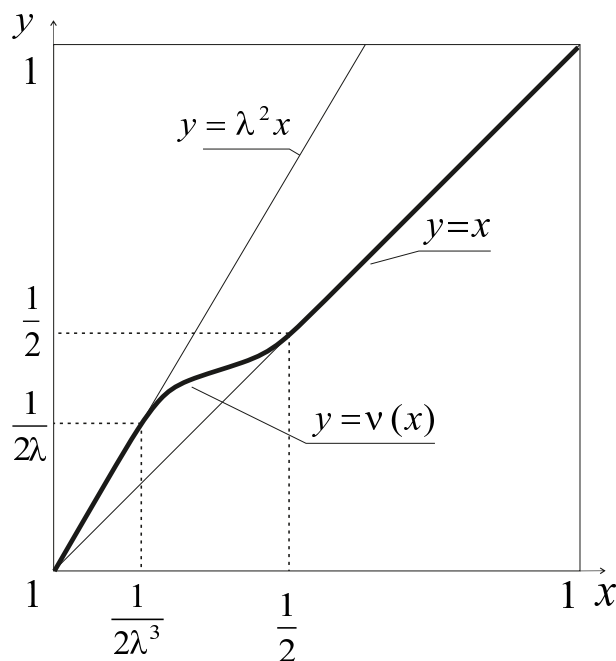
Рис. 1. Graph of funcion $\nu(x)$

By the construction B_t is an isotopy between the identity map $B_0(x, y)$ on $[0, 1]^2$ and $B_1(x, y) = B(x, y)$. As B preserves y-coordinate, we will use the following designation: $B(x, y) = (\gamma(x, y), y)$. Let $D = \{(x, y) \in U(O) : x^2 + y^2 \leqslant \frac{1}{(2\lambda^3)^2}\}$. By the construction B is identity out of $[0, 1/2]^2$ and for points $(x, y) \in D$ we have $\gamma(x, y) = \lambda^2 x$.

Let $\widehat{B}_t$ be a B_t inside $[0, 1]^2$ and is identity out of it. By arguments like to [7] it is possible to prove that $\widehat{\Psi} = \widehat{B} \circ \widehat{C}$ is a DA-diffeomorphism, whose non-wandering set consists of a one-dimensional attractor and a source.

2.3. One-dimensional surface attractor of $\mathbb{T}^2 \times \mathbb{R}$

Consider a smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ by the formula $\varphi(z) = \frac{z}{\lambda}$. Define a diffeomorphism of $\mathbb{T}^2 \times \mathbb{R}$ in coordinates $w \in \mathbb{T}^2, z \in \mathbb{R}$ by the formula

$$\Phi(w, z) = (\widehat{\Psi}(w), \varphi(z)).$$

The diffeomorphism Φ is an A-diffeomorphism whose non-wandering set contains one saddle point $\{O\} \times \{0\}$ and an one-dimensional attractor $\mathcal{A}$ which is placed on a 2-torus $\mathbb{T}^2 \times \{0\}$.

2.4. Fundamental domain of the attractor

To find a fundamental domain of the basin of the attractor $\mathcal{A}$, first of all, notice that $\mathbb{T}^2 \times (-\frac{1}{2\lambda^2}, \frac{1}{2\lambda^2})$ is a trapping neighbourhood of an attractor $\mathbb{T}^2 \times \{0\}$. It is not a trapping neighbourhood for $\mathcal{A}$ because there are a saddle point O and a segment of

its stable separatrix $\{O\} \times (-\frac{1}{2\lambda^2}, \frac{1}{2\lambda^2})$ inside. A small neighbourhood of the separatrix will be chosen to remove them.

So $U_1^{\mathcal{A}} = (\mathbb{T}^2 \setminus \widehat{\Psi}^{-1}(D)) \times \left(-\frac{1}{2\lambda^2}, \frac{1}{2\lambda^2}\right)$ is a desired trapping neighbourhood for the attractor $\mathcal{A}$. Indeed, $\Phi(U_1^{\mathcal{A}}) = (\mathbb{T}^2 \setminus D) \times \left(-\frac{1}{2\lambda^3}, \frac{1}{2\lambda^3}\right) = U_2^{\mathcal{A}} \subset int\, U_1^{\mathcal{A}}$ and $\bigcap\limits_{n=1}^{\infty} \Phi^n(U_1^{\mathcal{A}}) = \mathcal{A}$.
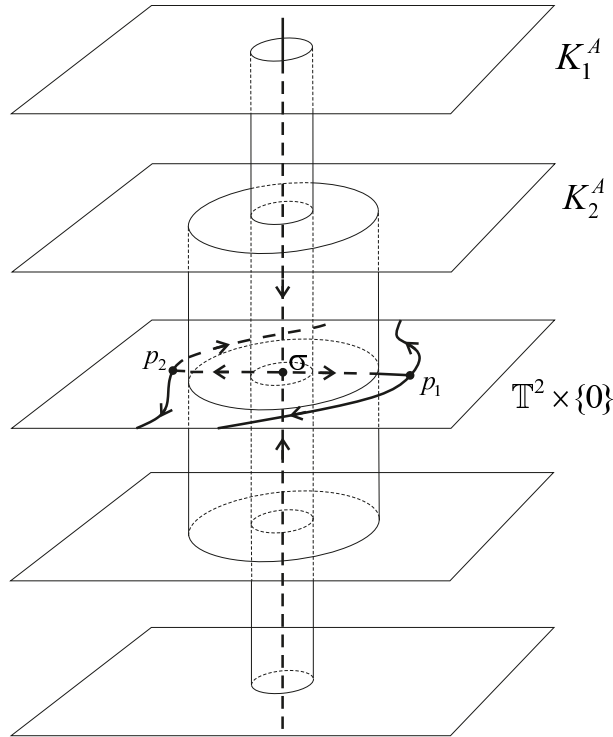

Рис. 2. Fundamental domain of an one-dimensional surface attractor

After that a fundamental domain of a basin of the attractor $\mathcal{A}$ is $K^{\mathcal{A}} = cl\,(U_1^{\mathcal{A}} \setminus U_2^{\mathcal{A}})$ (see Fig. 2). Notice, that $\partial U_1^{\mathcal{A}}$ is a pretzel and $K^{\mathcal{A}}$ is homeomorphic to the direct product of a pretzel by a segment. We will use the following designation: $K_1^{\mathcal{A}} = \partial U_1^{\mathcal{A}}$, $K_2^{\mathcal{A}} = \partial U_2^{\mathcal{A}}$. So $\Phi(K_1^{\mathcal{A}}) = K_2^{\mathcal{A}}$.

2.5. One-dimensional surface repeller of $\mathbb{T}^2 \times \mathbb{R}$

Define a diffeomorphism $\theta : [0,1]^2 \to [0,1]^2$ by the formula $\theta(x,y) = (-y,x)$. For every $t \in [0,1]$ let us define a diffeomorphism $Q_t : [0,1]^2 \to [0,1]^2$ by the formula $Q_t = \theta^{-1} B_t^{-1} \theta$. By the construction Q_t is an isotopy between the identity map $Q_0(x,y)$ on $[0,1]^2$ and $Q_1(x,y) = Q(x,y)$. Let $\widehat{Q}_t$ be a Q_t inside $[0,1]^2$ and is identity out of it. So $\widehat{Q}_t$ keeps stable foliation of $\widehat{C}$ and add additional contraction along unstable manifolds of $\widehat{C}$ inside some neighbourhood of O. Thus $\widehat{\Psi}_- = \widehat{Q} \circ \widehat{C}$ is a DA-diffeomorphism, whose non-wandering set consists of a one-dimensional repeller and a sink.

Consider a copy of $\mathbb{T}^2 \times \mathbb{R}$ with the following diffeomorphisms of it

$$\Phi_-(w, z) = (\widehat{\Psi}_-(w), \varphi^{-1}(z)).$$

The diffeomorphism Φ_- is an A-diffeomorphism whose non-wandering set contains one saddle point $\{O\} \times \{0\}$ and an one-dimensional repeller $\mathcal{R}$ which is placed on a 2-torus $\mathbb{T}^2 \times \{0\}$. Trapping neighbourhoods $U_1^{\mathcal{R}}$, $U_2^{\mathcal{R}}$ of $\mathcal{R}$ and a fundamental domain $K^{\mathcal{R}}$ of a basin of the repeller $\mathcal{R}$ are the same as for diffeomorphism Φ so $\Phi_-(U_2^{\mathcal{R}}) = U_1^{\mathcal{R}}$, $K_1^{\mathcal{R}} = \partial U_1^{\mathcal{R}}$, $K_2^{\mathcal{R}} = \partial U_2^{\mathcal{R}}$, and $\Phi_-(K_2^{\mathcal{R}}) = K_1^{\mathcal{R}}$.

2.6. Gluing of the fundamental domains

The goal of this section is to construct a diffeomorphism $H : K^{\mathcal{R}} \to K^{\mathcal{A}}$ with the properties:

- $H(K_2^{\mathcal{R}}) = H(K_1^{\mathcal{A}})$ and $H(K_1^{\mathcal{R}}) = H(K_2^{\mathcal{A}})$;

- $H = \Phi$ on $K_1^{\mathcal{R}}$ and $H = \Phi_-$ on $K_2^{\mathcal{R}}$.

For every $t \in [0, 1]$ let $\widehat{\xi}_t = \widehat{B}_t \circ \widehat{Q}_t^{-1} : \mathbb{T}^2 \to \mathbb{T}^2$. Notice that $\widehat{\xi}_1 = \widehat{\Psi} \circ \widehat{\Psi}_-^{-1}$. Then $\widehat{\xi}_t \circ \widehat{\Psi}_-$ is an isotopy between $\widehat{\Psi}_-$ and $\widehat{\Psi}$. By the construction $\widehat{\xi}_t \circ \widehat{\Psi}_-$ has a form $\widehat{\xi}_t \circ \widehat{\Psi}_-(x, y) = (k(t)x, k(t)y)$ for $(x, y) \in \widehat{\Psi}(D)$, where

$$k(t) = \left(\lambda - \frac{1}{\lambda}\right)t + \frac{1}{\lambda}.$$

Finally let $r, q : [0, 1] \to \left[\frac{1}{2\lambda^3}, \frac{1}{2\lambda^2}\right]$ be functions given by the formulas

$$r(t) = \frac{\lambda - 1}{2\lambda^3}t + \frac{1}{2\lambda^3}, \ \ q(t) = k(t) \cdot r(t).$$

Consider the fundamental domain $K^{\mathcal{R}}$ of the basin of the repeller $\mathcal{R}$ (all reasoning for the attractor $\mathcal{A}$ are analogical). Let $D_{r(t)} = \{(x, y) \in U(O) : x^2 + y^2 \leqslant r^2(t)\}$. Then $K^{\mathcal{R}}$ can be represented foliated by leaves

$$\{G_{r(t)} = G \times r(t), \ t \in [0, 1]\}$$

such that $G_{r(0)} = K_2^{\mathcal{R}}$, $G_{r(1)} = K_1^{\mathcal{R}}$ and $G_{r(t)}$ coincides with the tori $\mathbb{T}^2 \times \{\pm r(t)\}$ out of $D_{r(t)} \times \mathbb{R}$ and coincides with the cylinder $\partial D_{r(t)} \times [-r(t), r(t)]$ otherwise (see Fig. 3).

Define a map $H_t : G_{r(t)} \to G_{q(t)}$, $t \in [0, 1]$ as follows

$$H_t(w, z) = \left(\widehat{\xi}_t(\widehat{\Psi}_-(w)), \frac{q(t)}{r(t)}z\right).$$

Thus the diffeomorphism $H : K^{\mathcal{R}} \to K^{\mathcal{A}}$, composed by $H_t, t \in [0, 1]$, glues the dynamics of Φ_- and Φ along the fundamental domains. After a smoothing the corners
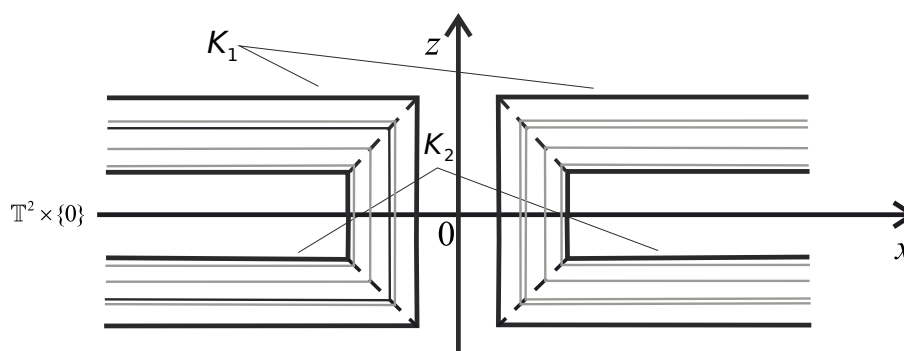
Рис. 3. Foliation of the fundamental domain

we get a new 3-manifold M with the desired A-diffeomorphism whose non-wandering set consists of one-dimensional surface repeller and one-dimensional surface attractor.

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