*Динамические системы, 2018, том 8(36), №1, 73–88*

MSC 2010: 34C05, 34C07, 34C23, 37G15

# Multi-parameter planar dynamical systems: global bifurcations of limit cycles[1]

**V. A. Gaiko***, **C. Vuik****

*United Institute of Informatics Problems, National Academy of Sciences of Belarus, Minsk 220012, Belarus. *E-mail: valery.gaiko@gmail.com*

**Institute of Applied Mathematics, Delft University of Technology, 2600 GA Delft, The Netherlands. *E-mail: c.vuik@tudelft.nl*

**Abstract.** We study multi-parameter planar dynamical systems and carry out the global bifurcation analysis of such systems. To control the global bifurcations of limit cycle in these systems, it is necessary to know the properties and combine the effects of all their field rotation parameters. It can be done by means of the development of our bifurcational geometric method based on the Wintner–Perko termination principle and application of canonical systems with field rotation parameters. Using this method, we solve, e. g., Hilbert's Sixteenth Problem on the maximum number of limit cycles and their distribution for the general Liénard polynomial system and a Holling-type quartic dynamical system. We also conduct some numerical experiments to illustrate the obtained results.

**Keywords:** multi-parameter planar dynamical system, global bifurcation analysis, field rotation parameter, limit cycle.

## 1. Introduction

We develop geometric aspects of bifurcation theory for studying multi-parameter planar polynomial dynamical systems. It gives a global approach to the qualitative analysis of such systems and helps to combine all other approaches, their methods and results. First of all, the two-isocline method which was developed by N. P. Erugin is used [4]. The isocline portrait is the most natural construction in the corresponding polynomial equation. It is sufficient to have only two isoclines (of zero and infinity) to obtain principal information on the original system, because these two isoclines are the right-hand sides of the system. Geometric properties of isoclines (conics, cubics, quartics, etc.) are well-known, and all isocline portraits can be easily constructed. By means of them, all topologically different qualitative pictures of integral curves to within a number of limit cycles and distinguishing center and focus can be obtained. Thus it is possible to carry out a rough topological classification of the phase portraits for the polynomial systems. It is the first application of Erugin's method. After studying contact and rotation properties of isoclines, the simplest (canonical) systems containing limit cycles can be also constructed. Two groups of parameters can be distinguished in such systems: static and dynamic. Static parameters determine the behavior of the

[1]The first author was supported by the Netherlands Organization for Scientific Research (NWO) and the Institute of Advanced Scientific Studies (IHÉS, France)

phase trajectories in principle, since they control the number, position and character of singular points in a finite part of the plane (finite singularities). Parameters from the first group determine also a possible behavior of separatrices and singular points at infinity (infinite singularities) under the variation of the parameters from the second group. Dynamic parameters are rotation parameters. They do not change the number, position and index of finite singularities and involve the vector field into directional rotation. The rotation parameters allow to control infinite singularities, the behavior of limit cycles and separatrices. The cyclicity of singular points and separatrix cycles, the behavior of semi-stable and other multiple limit cycles are controlled by these parameters as well. Therefore, by means of the rotation parameters, it is possible to control all limit cycle bifurcations and to solve the most complicated problems of the qualitative theory of polynomial systems [4].

To control all of the limit cycle bifurcations (especially, bifurcations of multiple limit cycles), it is necessary to know the properties and combine the effects of all of the rotation parameters. It can be done by means of the development of new methods based on the well-known Weierstrass preparation theorem and the Perko planar termination principle stating that the maximal one-parameter family of multiple limit cycles terminates either at a singular point, which is typically of the same multiplicity, or on a separatrix cycle, which is also typically of the same multiplicity [4, 14]. This principle is a consequence of the principle of natural termination which was stated for higher-dimensional dynamical systems by A. Wintner, who studied one-parameter families of periodic orbits of the restricted three-body problem and used Puiseux series to show that in the analytic case any one-parameter family of periodic orbits can be uniquely continued through any bifurcation except a period-doubling bifurcation. Such a bifurcation can happen, for example, in a Lorenz system. Besides, the periods in a one-parameter family of a higher-dimensional system can become unbounded in strange ways: for example, the periodic orbits may belong to a strange invariant set, strange attractor, generated at a bifurcation value for which there is a homoclinic tangency of the stable and unstable manifolds of the Poincaré map. This cannot happen for planar systems. That is why the Wintner–Perko termination principle is applied for studying multiple limit cycle bifurcations of the multi-parameter planar polynomial dynamical systems [4, 14].

We have already presented a solution of Hilbert's Sixteenth Problem in the quadratic case of polynomial systems proving that for quadratic systems four is really the maximum number of limit cycles and $(3:1)$ is their only possible distribution. The proof is carried out by contradiction applying catastrophe theory. On the first step, the non-existence of four limit cycles surrounding a singular point is proved. A canonical system containing three field-rotation parameters is considered and it is supposed that this system has four limit cycles around the origin. Thus we get into some three-dimensional domain of the field rotation parameters being restricted by some conditions on the rest two parameters corresponding to definite cases of singular points in the phase plane. This three-parameter domain of four limit cycles is bounded by three fold bifurcation surfaces forming a swallow-tail bifurcation surface of multiplicity-four limit cycles. Since the corresponding maximal one-parameter family of multiplicity-four limit cycles generated by a field rotation is monotonic, it is proved that it cannot be cyclic and terminates either at the origin or on some separatrix cycle surrounding the origin. Besides, we know absolutely precisely the cyclicity of the singular point which is equal to three and therefore we have got a contradiction with the termination principle stating that the multiplicity of limit cycles cannot be higher than the multiplicity (cyclicity) of the singular point in which they terminate. Since we know the concrete properties of all three field rotation parameters in the canonical system and can control simultaneously bifurcations of limit cycles around different singular points, we are able to complete the proof of the theorem [4]. The same result can be obtained by purely geometric methods as well [6].

We have also established some preliminary results on generalizing our ideas and methods to special planar cubic, quartic and other polynomial dynamical systems. In [5], we have constructed a canonical cubic dynamical system of Kukles type and have carried out the global qualitative analysis of its special case corresponding to a generalized Liénard equation. In [11, 12], using the Wintner–Perko termination principle of multiple limit cycles and our bifurcational geometric approach, we have solved the problem on the maximum number and distribution of limit cycles in the general Kukles cubic-linear system. In [2], we have established the global qualitative analysis of centrally symmetric cubic systems which are used as learning models of planar neural networks. In [3], we have carried out the global bifurcation analysis of a quartic dynamical system which models the dynamics of the populations of predators and their prey in a given ecological system. We have also completed the study of multiple limit cycle bifurcations in the well-known FitzHugh–Nagumo neuronal model [7]. Besides, we have presented a solution of Smale's Thirteenth Problem [15] proving that the Liénard system with a polynomial of degree $2k + 1$ can have at most $k$ limit cycles [8]. Generalizing the obtained results, we have presented a solution of Hilbert's Sixteenth Problem on the maximum number of limit cycles surrounding a singular point for an arbitrary polynomial system [8].

In Section 2 of this paper, applying a canonical system with field rotation parameters and using geometric properties of the spirals filling the interior and exterior domains of limit cycles, we solve the limit cycle problem for the general Liénard polynomial system with an arbitrary (but finite) number of singular points generalizing our previous results which we obtained in [9, 10] under some assumptions on the parameters of the Liénard system. In Section 3, we complete the global bifurcation analysis of a quartic dynamical system corresponding to a new class of rational Holling-type systems which model the dynamics of the populations of predators and their prey in a given ecological or biomedical system. We also conduct some numerical experiments to illustrate the results obtained in this paper [16].

## 2. The General Liénard Polynomial System

In this Section, by means of our bifurcational geometric approach [2]–[10], we consider the general Liénard polynomial system:

$$\dot{x} = y, \quad \dot{y} = -x\,(1 + a_1\,x + \ldots + a_{2l}\,x^{2l}) + y\,(\alpha_0 + \alpha_1\,x + \ldots + \alpha_{2k}\,x^{2k}). \tag{2.1}$$

Suppose that $a_1^2 + \ldots + a_{2l}^2 \neq 0$ in system (2.1). The finite singularities of (2.1) are determined by the algebraic system

$$x\,(1 + a_1\,x + \ldots + a_{2l}\,x^{2l}) = 0, \quad y = 0. \tag{2.2}$$

This system always has an anti-saddle at the origin and, in general, can have at most $2l + 1$ finite singularities which lie on the $x$-axis and are distributed so that a saddle (or saddle-node) is followed by a node or a focus, or a center and vice versa [1]. For studying the infinite singularities, the methods applied in [1] for Rayleigh's and van der Pol's equations and also Erugin's two-isocline method developed in [4] can be used.

Following [4], we will study limit cycle bifurcations of (2.1) by means of canonical systems containing field rotation parameters of (2.1) [1, 4].

**Theorem 2.1.** *The Liénard polynomial system* (2.1) *with limit cycles can be reduced to one of the canonical forms:*

$$\begin{gathered}\dot{x} = y,\\ \dot{y} = -x\,(1 + a_1 x + \ldots + a_{2l} x^{2l})\\ + y(\alpha_0 - \beta_1 - \ldots - \beta_{2k-1} + \beta_1 x + \alpha_2 x^2 + \ldots + \beta_{2k-1} x^{2k-1} + \alpha_{2k} x^{2k})\end{gathered} \tag{2.3}$$

*or*

$$\begin{gathered}\dot{x} = y \equiv P(x,y),\\ \dot{y} = x(x-1)(1 + b_1 x + \ldots + b_{2l-1} x^{2l-1})\\ + y(\alpha_0 - \beta_1 - \ldots - \beta_{2k-1} + \beta_1 x + \alpha_2 x^2 + \ldots + \beta_{2k-1} x^{2k-1} + \alpha_{2k} x^{2k}) \equiv Q(x,y),\end{gathered} \tag{2.4}$$

*where* $1 + a_1 x + \ldots + a_{2l} x^{2l} \neq 0$, $\alpha_0, \alpha_2, \ldots, \alpha_{2k}$ *are field rotation parameters and* $\beta_1, \beta_3, \ldots, \beta_{2k-1}$ *are semi-rotation parameters.*

*Proof.* Let us compare system (2.1) with (2.3) and (2.4). It is easy to see that system (2.3) has the only finite singular point: an anti-saddle at the origin. System (2.4) has at list two singular points including an anti-saddle at the origin and a saddle which, without loss of generality, can be always putted into the point $(1, 0)$. Instead of the odd parameters $\alpha_1, \alpha_3, \ldots, \alpha_{2k-1}$ in system (2.1), also without loss of generality, we have introduced new parameters $\beta_1, \beta_3, \ldots, \beta_{2k-1}$ into (2.3) and (2.4).

We will study now system (2.4) (system (2.3) can be studied absolutely similarly). Let all of the parameters $\alpha_0, \alpha_2, \ldots, \alpha_{2k}$ and $\beta_1, \beta_3, \ldots, \beta_{2k-1}$ vanish in this system,

$$\dot{x} = y, \quad \dot{y} = x(x-1)(1 + b_1 x + \ldots + b_{2l-1} x^{2l-1}), \tag{2.5}$$

and consider the corresponding equation

$$\frac{dy}{dx} = \frac{x(x-1)(1+b_1x+\ldots+b_{2l-1}x^{2l-1})}{y} \equiv F(x,y). \tag{2.6}$$

Since $F(x,-y) = -F(x,y)$, the direction field of (2.6) (and the vector field of (2.5) as well) is symmetric with respect to the $x$-axis. It follows that for arbitrary values of the parameters $b_1, \ldots, b_{2l-1}$ system (2.5) has centers as anti-saddles and cannot have limit cycles surrounding these points. Therefore, we can fix the parameters $b_1, \ldots, b_{2l-1}$ in system (2.4), fixing the position of its finite singularities on the $x$-axis.

To prove that the even parameters $\alpha_0$, $\alpha_2, \ldots,$ $\alpha_{2k}$ rotate the vector field of (2.4), let us calculate the following determinants:

$$\Delta_{\alpha_0} = P\,Q'_{\alpha_0} - QP'_{\alpha_0} = y^2 \geq 0,$$
$$\Delta_{\alpha_2} = P\,Q'_{\alpha_2} - QP'_{\alpha_2} = x^2y^2 \geq 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\Delta_{\alpha_{2k}} = P\,Q'_{\alpha_{2k}} - QP'_{\alpha_{2k}} = x^{2k}y^2 \geq 0.$$

By definition of a field rotation parameter [1, 4, 14], for increasing each of the parameters $\alpha_0$, $\alpha_2, \ldots,$ $\alpha_{2k}$, under the fixed others, the vector field of system (2.4) is rotated in the positive direction (counterclockwise) in the whole phase plane; and, conversely, for decreasing each of these parameters, the vector field of (2.4) is rotated in the negative direction (clockwise).

Calculating the corresponding determinants for the parameters $\beta_1$, $\beta_3, \ldots,$ $\beta_{2k-1}$, we can see that

$$\Delta_{\beta_1} = P\,Q'_{\beta_1} - QP'_{\beta_1} = (x-1)\,y^2,$$
$$\Delta_{\beta_3} = P\,Q'_{\beta_3} - QP'_{\beta_3} = (x^3-1)\,y^2,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\Delta_{\beta_{2k-1}} = P\,Q'_{\beta_{2k-1}} - QP'_{\beta_{2k-1}} = (x^{2k-1}-1)\,y^2.$$

It follows [1, 4] that, for increasing each of the parameters $\beta_1$, $\beta_3, \ldots,$ $\beta_{2k-1}$, under the fixed others, the vector field of system (2.4) is rotated in the positive direction (counterclockwise) in the half-plane $x > 1$ and in the negative direction (clockwise) in the half-plane $x < 1$ and vice versa for decreasing each of these parameters. We will call these parameters as semi-rotation ones.

Thus, for studying limit cycle bifurcations of (2.1), it is sufficient to consider the canonical systems (2.3) and (2.4) containing the field rotation parameters $\alpha_0$, $\alpha_2, \ldots,$ $\alpha_{2k}$ and the semi-rotation parameters $\beta_1$, $\beta_3, \ldots,$ $\beta_{2k-1}$. The theorem is proved. □

By means of the canonical systems (2.3) and (2.4), we will prove the following theorem.

**Theorem 2.2.** *The Liénard polynomial system* (2.1) *can have at most* $k + l + 1$ *limit cycles,* $k+1$ *surrounding the origin and* $l$ *surrounding one by one the other singularities of* (2.1).

*Proof.* According to Theorem 2.1, for the study of limit cycle bifurcations of system (2.1), it is sufficient to consider the canonical systems (2.3) and (2.4) containing the field rotation parameters $\alpha_0, \ \alpha_2, \ldots, \ \alpha_{2k}$ and the semi-rotation parameters $\beta_1, \ \beta_3, \ldots, \ \beta_{2k-1}$. We will work with (2.4) again (system (2.3) can be considered in a similar way).

Vanishing all of the parameters $\alpha_0, \ \alpha_2, \ldots, \ \alpha_{2k}$ and $\beta_1, \ \beta_3, \ldots, \ \beta_{2k-1}$ in (2.4), we will have system (2.5) which is symmetric with respect to the $x$-axis and has centers as anti-saddles. Its center domains are bounded by either separatrix loops or digons of the saddles or saddle-nodes of (2.5) lying on the $x$-axis.

Let us input successively the semi-rotation parameters $\beta_1, \ \beta_3, \ldots, \ \beta_{2k-1}$ into system (2.5) beginning with the parameters at the highest degrees of $x$ and alternating with their signs. So, begin with the parameter $\beta_{2k-1}$ and let, for definiteness, $\beta_{2k-1} > 0$:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x-1)(1 + b_1 x + \ldots + b_{2l-1}x^{2l-1}) \\ &\quad + y(-\beta_{2k-1} + \beta_{2k-1}x^{2k-1}). \end{aligned} \tag{2.7}$$

In this case, the vector field of (2.7) is rotated in the negative direction (clockwise) in the half-plane $x < 1$ turning the center at the origin into a rough stable focus. All of the other centers lying in the half-plane $x > 1$ become rough unstable foci, since the vector field of (2.7) is rotated in the positive direction (counterclockwise) in this half-plane [1, 4].

Fix $\beta_{2k-1}$ and input the parameter $\beta_{2k-3} < 0$ into (2.7):

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x-1)(1 + b_1 x + \ldots + b_{2l-1}x^{2l-1}) \\ &\quad + y(-\beta_{2k-3} - \beta_{2k-1} + \beta_{2k-3}x^{2k-3} + \beta_{2k-1}x^{2k-1}). \end{aligned} \tag{2.8}$$

Then the vector field of (2.8) is rotated in the opposite directions in each of the half-planes $x < 1$ and $x > 1$. Under decreasing $\beta_{2k-3}$, when $\beta_{2k-3} = -\beta_{2k-1}$, the focus at the origin becomes nonrough (weak), changes the character of its stability and generates a stable limit cycle. All of the other foci in the half-plane $x > 1$ will also generate unstable limit cycles for some values of $\beta_{2k-3}$ after changing the character of their stability. Under further decreasing $\beta_{2k-3}$, all of the limit cycles will expand disappearing on separatrix cycles of (2.8) [1, 4].

Denote the limit cycle surrounding the origin by $\Gamma_0$, the domain outside the cycle by $D_{01}$, the domain inside the cycle by $D_{02}$ and consider logical possibilities of the appearance of other (semi-stable) limit cycles from a "trajectory concentration" surrounding this singular point. It is clear that, under decreasing the parameter $\beta_{2k-3}$, a semi-stable limit cycle cannot appear in the domain $D_{02}$, since the focus spirals filling this domain will untwist and the distance between their coils will increase because of the vector field rotation [4].

By contradiction, we can also prove that a semi-stable limit cycle cannot appear in the domain $D_{01}$. Suppose it appears in this domain for some values of the parameters $\beta^*_{2k-1} > 0$ and $\beta^*_{2k-3} < 0$. Return to system (2.5) and change the inputting order for the semi-rotation parameters. Input first the parameter $\beta_{2k-3} < 0$:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x-1)(1+b_1x+\ldots+b_{2l-1}x^{2l-1}) \\ &\quad + y(-\beta_{2k-3}+\beta_{2k-3}x^{2k-3}). \end{aligned} \tag{2.9}$$

Fix it under $\beta_{2k-3} = \beta^*_{2k-3}$. The vector field of (2.9) is rotated counterclockwise and the origin turns into a rough unstable focus. Inputting the parameter $\beta_{2k-1} > 0$ into (2.9), we get again system (2.8) the vector field of which is rotated clockwise. Under this rotation, a stable limit cycle $\Gamma_0$ will appear from a separatrix cycle for some value of $\beta_{2k-1}$. This cycle will contract, the outside spirals winding onto the cycle will untwist and the distance between their coils will increase under increasing $\beta_{2k-1}$ to the value $\beta^*_{2k-1}$. It follows that there are no values of $\beta^*_{2k-3} < 0$ and $\beta^*_{2k-1} > 0$ for which a semi-stable limit cycle could appear in the domain $D_{01}$.

This contradiction proves the uniqueness of a limit cycle surrounding the origin in system (2.8) for any values of the parameters $\beta_{2k-3}$ and $\beta_{2k-1}$ of different signs. Obviously, if these parameters have the same sign, system (2.8) has no limit cycles surrounding the origin at all. On the same reason, this system cannot have more than $l$ limit cycles surrounding the other singularities (foci or nodes) of (2.8) one by one.

It is clear that inputting the other semi-rotation parameters $\beta_{2k-5}, \ldots, \beta_1$ into system (2.8) will not give us more limit cycles, since all of these parameters are rough with respect to the origin and the other anti-saddles lying in the half-plane $x > 1$. Therefore, the maximum number of limit cycles for the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x-1)(1+b_1x+\ldots+b_{2l-1}x^{2l-1}) \\ &\quad + y(-\beta_1-\ldots-\beta_{2k-3}-\beta_{2k-1}+\beta_1x+\ldots+\beta_{2k-3}x^{2k-3}+\beta_{2k-1}x^{2k-1}) \end{aligned} \tag{2.10}$$

is equal to $l+1$ and they surround the anti-saddles (foci or nodes) of (2.10) one by one.

Suppose that $\beta_1+\ldots+\beta_{2k-3}+\beta_{2k-1} > 0$ and input the last rough parameter $\alpha_0 > 0$ into system (2.10):

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x-1)(1+b_1x+\ldots+b_{2l-1}x^{2l-1}) \\ &\quad + y(\alpha_0-\beta_1-\ldots-\beta_{2k-1}+\beta_1x+\ldots+\beta_{2k-1}x^{2k-1}). \end{aligned} \tag{2.11}$$

This parameter rotating the vector field of (2.11) counterclockwise in the whole phase plane also will not give us more limit cycles, but under increasing $\alpha_0$, when $\alpha_0 =$

$\beta_1 + \ldots + \beta_{2k-1}$, we can make the focus at the origin nonrough (weak), after the disappearance of the limit cycle $\Gamma_0$ in it. Fix this value of the parameter $\alpha_0$ ($\alpha_0 = \alpha_0^*$):

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= x(x-1)(1 + b_1 x + \ldots + b_{2l-1}x^{2l-1})\\ &\quad + y(\beta_1 x + \ldots + \beta_{2k-1}x^{2k-1}).\end{aligned} \tag{2.12}$$

Let us input now successively the other field rotation parameters $\alpha_2, \ldots, \alpha_{2k}$ into system (2.12) beginning again with the parameters at the highest degrees of $x$ and alternating with their signs. So, begin with the parameter $\alpha_{2k}$ and let $\alpha_{2k} < 0$:

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= x(x-1)(1 + b_1 x + \ldots + b_{2l-1}x^{2l-1})\\ &\quad + y(\beta_1 x + \ldots + \beta_{2k-1}x^{2k-1} + \alpha_{2k}x^{2k}).\end{aligned} \tag{2.13}$$

In this case, the vector field of (2.13) is rotated clockwise in the whole phase plane and the focus at the origin changes the character of its stability generating again a stable limit cycle. The limit cycles surrounding the other singularities of (2.13) can also still exist. Denote the limit cycle surrounding the origin by $\Gamma_1$, the domain outside the cycle by $D_1$ and the domain inside the cycle by $D_2$. The uniqueness of a limit cycle surrounding the origin (and limit cycles surrounding the other singularities) for system (2.13) can be proved by contradiction like we have done above for (2.8).

Let system (2.13) have the unique limit cycle $\Gamma_1$ surrounding the origin and $l$ limit cycles surrounding the other antisaddles of (2.13). Fix the parameter $\alpha_{2k} < 0$ and input the parameter $\alpha_{2k-2} > 0$ into (2.13):

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= x(x-1)(1 + b_1 x + \ldots + b_{2l-1}x^{2l-1})\\ &\quad + y(\beta_1 x + \ldots + \beta_{2k-1}x^{2k-1} + \alpha_{2k-2}x^{2k-2} + \alpha_{2k}x^{2k}).\end{aligned} \tag{2.14}$$

Then the vector field of (2.14) is rotated in the opposite direction (counterclockwise) and the focus at the origin immediately changes the character of its stability (since its degree of nonroughness decreases and the sign of the field rotation parameter at the lower degree of $x$ changes) generating the second (unstable) limit cycle $\Gamma_2$. The limit cycles surrounding the other singularities of (2.14) can only disappear in the corresponding foci (because of their roughness) under increasing the parameter $\alpha_{2k-2}$. Under further increasing $\alpha_{2k-2}$, the limit cycle $\Gamma_2$ will join with $\Gamma_1$ forming a semi-stable limit cycle, $\Gamma_{12}$, which will disappear in a "trajectory concentration" surrounding the origin. Can another semi-stable limit cycle appear around the origin in addition to $\Gamma_{12}$? It is clear that such a limit cycle cannot appear either in the domain $D_1$ bounded on the inside by the cycle $\Gamma_1$ or in the domain $D_3$ bounded by the origin and $\Gamma_2$ because of the increasing distance between the spiral coils filling these domains under increasing the parameter.

To prove the impossibility of the appearance of a semi-stable limit cycle in the domain $D_2$ bounded by the cycles $\Gamma_1$ and $\Gamma_2$ (before their joining), suppose the contrary,

i. e., that for some values of these parameters, $\alpha^*_{2k} < 0$ and $\alpha^*_{2k-2} > 0$, such a semi-stable cycle exists. Return to system (2.12) again and input first the parameter $\alpha_{2k-2} > 0$:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x-1)(1 + b_1 x + \ldots + b_{2l-1}x^{2l-1}) \\ &+ y(\beta_1 x + \ldots + \beta_{2k-1}x^{2k-1} + \alpha_{2k-2}x^{2k-2}). \end{aligned} \tag{2.15}$$

This parameter rotates the vector field of (2.15) counterclockwise preserving the origin as a nonrough stable focus.

Fix this parameter under $\alpha_{2k-2} = \alpha^*_{2k-2}$ and input the parameter $\alpha_{2k} < 0$ into (2.15) getting again system (2.14). Since, by our assumption, this system has two limit cycles surrounding the origin for $\alpha_{2k} > \alpha^*_{2k}$, there exists some value of the parameter, $\alpha^{12}_{2k}$ ($\alpha^{12}_{2k} < \alpha^*_{2k} < 0$), for which a semi-stable limit cycle, $\Gamma_{12}$, appears in system (2.14) and then splits into a stable cycle $\Gamma_1$ and an unstable cycle $\Gamma_2$ under further decreasing $\alpha_{2k}$. The formed domain $D_2$ bounded by the limit cycles $\Gamma_1$, $\Gamma_2$ and filled by the spirals will enlarge since, on the properties of a field rotation parameter, the interior unstable limit cycle $\Gamma_2$ will contract and the exterior stable limit cycle $\Gamma_1$ will expand under decreasing $\alpha_{2k}$. The distance between the spirals of the domain $D_2$ will naturally increase, which will prevent the appearance of a semi-stable limit cycle in this domain for $\alpha_{2k} < \alpha^{12}_{2k}$.

Thus, there are no such values of the parameters, $\alpha^*_{2k} < 0$ and $\alpha^*_{2k-2} > 0$, for which system (2.14) would have an additional semi-stable limit cycle surrounding the origin. Obviously, there are no other values of the parameters $\alpha_{2k}$ and $\alpha_{2k-2}$ for which system (2.14) would have more than two limit cycles surrounding this singular point. On the same reason, additional semi-stable limit cycles cannot appear around the other singularities (foci or nodes) of (2.14). Therefore, $l + 2$ is the maximum number of limit cycles in system (2.14).

Suppose that system (2.14) has two limit cycles, $\Gamma_1$ and $\Gamma_2$, surrounding the origin and $l$ limit cycles surrounding the other antisaddles of (2.14) (this is always possible if $-\alpha_{2k} \gg \alpha_{2k-2} > 0$). Fix the parameters $\alpha_{2k}$, $\alpha_{2k-2}$ and consider a more general system inputting the third parameter, $\alpha_{2k-4} < 0$, into (2.14):

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x-1)(1 + b_1 x + \ldots + b_{2l-1}x^{2l-1}) \\ &+ y(\beta_1 x + \ldots + \beta_{2k-1}x^{2k-1} + \alpha_{2k-4}x^{2k-4} + \alpha_{2k-2}x^{2k-2} + \alpha_{2k}x^{2k}). \end{aligned} \tag{2.16}$$

For decreasing $\alpha_{2k-4}$, the vector field of (2.16) will be rotated clockwise and the focus at the origin will immediately change the character of its stability generating a third (stable) limit cycle, $\Gamma_3$. With further decreasing $\alpha_{2k-4}$, $\Gamma_3$ will join with $\Gamma_2$ forming a semi-stable limit cycle, $\Gamma_{23}$, which will disappear in a "trajectory concentration" surrounding the origin; the cycle $\Gamma_1$ will expand disappearing on a separatrix cycle of (2.16).

Let system (2.16) have three limit cycles surrounding the origin: $\Gamma_1$, $\Gamma_2$, $\Gamma_3$. Could an additional semi-stable limit cycle appear with decreasing $\alpha_{2k-4}$ after splitting of which system (2.16) would have five limit cycles around the origin? It is clear that such a limit cycle cannot appear either in the domain $D_2$ bounded by the cycles $\Gamma_1$ and $\Gamma_2$ or in the domain $D_4$ bounded by the origin and $\Gamma_3$ because of the increasing distance between the spiral coils filling these domains after decreasing $\alpha_{2k-4}$. Consider two other domains: $D_1$ bounded on the inside by the cycle $\Gamma_1$ and $D_3$ bounded by the cycles $\Gamma_2$ and $\Gamma_3$. As before, we will prove the impossibility of the appearance of a semi-stable limit cycle in these domains by contradiction.

Suppose that for some set of values of the parameters $\alpha^*_{2k} < 0$, $\alpha^*_{2k-2} > 0$ and $\alpha^*_{2k-4} < 0$ such a semi-stable cycle exists. Return to system (2.12) again inputting first the parameters $\alpha_{2k-2} > 0$ and $\alpha_{2k-4} < 0$:

$$\begin{gathered}\dot{x} = y,\\ \dot{y} = x(x-1)(1 + b_1 x + \ldots + b_{2l-1} x^{2l-1})\\ + y(\beta_1 x + \ldots + \beta_{2k-1} x^{2k-1} + \alpha_{2k-4} x^{2k-4} + \alpha_{2k} x^{2k}).\end{gathered} \tag{2.17}$$

Fix the parameter $\alpha_{2k-2}$ under the value $\alpha^*_{2k-2}$. With decreasing $\alpha_{2k-4}$, a separatrix cycle formed around the origin will generate a stable limit cycle $\Gamma_1$. Fix $\alpha_{2k-4}$ under the value $\alpha^*_{2k-4}$ and input the parameter $\alpha_{2k} > 0$ into (2.17) getting system (2.16).

Since, by our assumption, (2.16) has three limit cycles for $\alpha_{2k} > \alpha^*_{2k}$, there exists some value of the parameter $\alpha^{23}_{2k}$ ($\alpha^{23}_{2k} < \alpha^*_{2k} < 0$) for which a semi-stable limit cycle, $\Gamma_{23}$, appears in this system and then splits into an unstable cycle $\Gamma_2$ and a stable cycle $\Gamma_3$ with further decreasing $\alpha_{2k}$. The formed domain $D_3$ bounded by the limit cycles $\Gamma_2$, $\Gamma_3$ and also the domain $D_1$ bounded on the inside by the limit cycle $\Gamma_1$ will enlarge and the spirals filling these domains will untwist excluding a possibility of the appearance of a semi-stable limit cycle there.

All other combinations of the parameters $\alpha_{2k}$, $\alpha_{2k-2}$, and $\alpha_{2k-4}$ are considered in a similar way. It follows that system (2.16) can have at most $l + 3$ limit cycles.

If we continue the procedure of successive inputting the field rotation parameters, $\alpha_{2k}, \ldots, \alpha_2$, into system (2.12),

$$\begin{gathered}\dot{x} = y,\\ \dot{y} = x(x-1)(1 + b_1 x + \ldots + b_{2l-1} x^{2l-1})\\ + y(\beta_1 x + \ldots + \beta_{2k-1} x^{2k-1} + \alpha_2 x^2 + \ldots + \alpha_{2k} x^{2k}),\end{gathered} \tag{2.18}$$

it is possible to obtain $k$ limit cycles surrounding the origin and $l$ surrounding one by one the other singularities (foci or nodes) ($-\alpha_{2k} \gg \alpha_{2k-2} \gg -\alpha_{2k-4} \gg \alpha_{2k-6} \gg \ldots$).

Then, by means of the parameter $\alpha_0 \neq \beta_1 + \ldots + \beta_{2k-1}$ ($\alpha_0 > \alpha^*_0$, if $\alpha_2 < 0$, and $\alpha_0 < \alpha^*_0$, if $\alpha_2 > 0$), we will have the canonical system (2.4) with an additional limit cycle surrounding the origin and can conclude that this system (i. e., the Liénard polynomial system (2.1) as well) has at most $k+l+1$ limit cycles, $k+1$ surrounding the origin and $l$ surrounding one by one the antisaddles (foci or nodes) of (2.4) (and (2.1) as well). The theorem is proved. □

## 3. A Holling-Type Quartic Dynamical System

In this Section, we study a Holling-type rational system which models the dynamics of the populations of predators and their prey in a given ecological or biomedical system:

$$\begin{aligned} \dot{x} &= x\left(1 - \lambda x - \frac{xy}{\alpha x^2 + \beta x + 1}\right) \qquad \text{(prey)}, \\ \dot{y} &= -y\left(\delta + \mu y - \frac{x^2}{\alpha x^2 + \beta x + 1}\right) \quad \text{(predator)}, \end{aligned} \tag{3.1}$$

where $x > 0$ and $y > 0$; $\alpha \geq 0$, $\beta > -2\sqrt{\alpha}$, $\delta > 0$, $\lambda > 0$, and $\mu \geq 0$ are parameters.

Dividing the second equation of (3.1) by the first one (left and right hand sides, respectively), after algebraic transformations in the corresponding equation, we can rewrite rational system (3.1) in the form of a quartic dynamical system

$$\begin{aligned} \dot{x} &= \quad x((1 - \lambda x)(\alpha x^2 + \beta x + 1) - xy) \equiv P, \\ \dot{y} &= -y((\delta + \mu y)(\alpha x^2 + \beta x + 1) - x^2) \equiv Q. \end{aligned} \tag{3.2}$$

Together with (3.2), we will also consider an auxiliary system; see [1, 4, 14]

$$\dot{x} = P - \gamma Q, \qquad \dot{y} = Q + \gamma P, \tag{3.3}$$

applying to these systems our bifurcational geometric approach [2]–[10] and completing the qualitative analysis of (3.1).

Consider first a general polynomial system in the vector form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\mu}), \tag{3.4}$$

where $\boldsymbol{x} \in \mathbf{R}^2$; $\boldsymbol{\mu} \in \mathbf{R}^n$; $\boldsymbol{f} \in \mathbf{R}^2$ ( $\boldsymbol{f}$ is a polynomial vector function).

Let us formulate the Wintner–Perko termination principle [4, 14] for this system.

**Theorem 3.1.** *Any one-parameter family of multiplicity-m limit cycles of relatively prime polynomial system* (3.4) *can be extended in a unique way to a maximal one-parameter family of multiplicity-m limit cycles of* (3.4) *which is either open or cyclic.*

*If it is open, then it terminates either as the parameter or the limit cycles become unbounded; or, the family terminates either at a singular point of* (3.4)*, which is typically a fine focus of multiplicity m, or on a (compound) separatrix cycle of* (3.4) *which is also typically of multiplicity m.*

The proof of this principle for general polynomial system (3.4) with a vector parameter $\boldsymbol{\mu} \in \mathbf{R}^n$ parallels the proof of the planar termination principle for the system

$$\dot{x} = P(x, y, \lambda), \quad \dot{y} = Q(x, y, \lambda) \tag{3.5}$$

with a single parameter $\lambda \in \mathbf{R}$ [4, 14], since there is no loss of generality in assuming that system (3.4) is parameterized by a single parameter $\lambda$; i. e., we can assume that there exists an analytic mapping $\boldsymbol{\mu}(\lambda)$ of $\mathbf{R}$ into $\mathbf{R}^n$ such that (3.4) can be written as (3.5) and then we can repeat everything, what had been done for system (3.5) in [14]. In particular, if $\lambda$ is a field rotation parameter of (3.5), the following Perko's theorem on monotonic families of limit cycles is valid [4, 14].

**Theorem 3.2.** *If $L_0$ is a nonsingular multiple limit cycle of* (3.5) *for $\lambda = \lambda_0$, then $L_0$ belongs to a one-parameter family of limit cycles of* (3.5)*; furthermore:*

1) *if the multiplicity of $L_0$ is odd, then the family either expands or contracts monotonically as $\lambda$ increases through $\lambda_0$;*

2) *if the multiplicity of $L_0$ is even, then $L_0$ bifurcates into a stable and an unstable limit cycle as $\lambda$ varies from $\lambda_0$ in one sense and $L_0$ disappears as $\lambda$ varies from $\lambda_0$ in the opposite sense; i. e., there is a fold bifurcation at $\lambda_0$.*

Consider again system (3.2). This system has two invariant straight lines: $x = 0$ and $y = 0$. Its finite singularities are determined by the algebraic system

$$\begin{aligned} &x((1-\lambda x)(\alpha x^2+\beta x+1)-xy)=0,\\ &y((\delta+\mu y)(\alpha x^2+\beta x+1)-x^2)=0. \end{aligned} \tag{3.6}$$

From (3.6), we have got: two singular points $(0,0)$ and $(0,-\delta/\mu)$, at most two points defined by the condition

$$\alpha x^2+\beta x+1=0, \quad y=0, \tag{3.7}$$

and at most six singularities defined by the system

$$\begin{aligned} &xy=(1-\lambda x)(\alpha x^2+\beta x+1),\\ &y\,(\delta+\mu y)=x\,(1-\lambda x), \end{aligned} \tag{3.8}$$

among which we always have the point $(1/\lambda, 0)$. See [13] for more details.

The point $(0,0)$ is always a saddle, but $(1/\lambda, 0)$ can be a node or a saddle, or a saddle-node. The point $(1/\lambda, 0)$ can change multiplicity when singular points enter or exit the first quadrant. In addition, a singular point of multiplicity 2 may appear in the first quadrant and bifurcate into two singular points. In the case $\beta \ge 0$ (respectively, $-2\sqrt{\alpha} < \beta < 0$), there is a possibility of up to one singular point (respectively, two singular points) in the open first quadrant [13]. If there exists exactly one simple singular point in the open first quadrant, then it is an anti-saddle. If there exists exactly two simple singular points in the open first quadrant, then the singular point on the left with respect to the $x$-axis is an anti-saddle and the singular point on the right is a saddle [13]. If a singular point is not in the first quadrant, in consequence, it has no biological significance.

To study singular points of (3.2) at infinity, consider the corresponding differential equation

$$\frac{dy}{dx}=-\frac{y((\delta+\mu y)(\alpha x^2+\beta x+1)-x^2)}{x((1-\lambda x)(\alpha x^2+\beta x+1)-xy)}. \tag{3.9}$$

Dividing the numerator and denominator of the right-hand side of (3.9) by $x^4$ ($x \neq 0$) and denoting $y/x$ by $u$ (as well as $dy/dx$), we will get the algebraic equation

$$u((\mu/\lambda)u - 1) = 0, \quad \text{where} \quad u = y/x, \tag{3.10}$$

for all infinite singularities of (3.9) except when $x = 0$ (the "ends" of the $y$-axis) [1, 4]. For this special case we can divide the numerator and denominator of the right-hand side of (3.9) by $y^4$ ($y \neq 0$) denoting $x/y$ by $v$ (as well as $dx/dy$) and consider the algebraic equation

$$v^3(v - \mu/\lambda) = 0, \quad \text{where} \quad v = x/y. \tag{3.11}$$

The equations (3.10) and (3.11) give three singular points at infinity for (3.9): a simple node on the "ends" of the $x$-axis, a triple node on the "ends" of the $y$-axis, and a simple saddle in the direction of $y/x = \lambda/\mu$.

To investigate the character and distribution of the singular points in the phase plane, we have used a method developed in [3]. The sense of this method is to obtain the simplest (well-known) system by vanishing some parameters (usually field rotation parameters) of the original system and then to input these parameters successively one by one studying the dynamics of the singular points (both finite and infinite) in the phase plane.

Using the obtained information on singular points and applying our bifurcational geometric approach [2]–[10], we can study the limit cycle bifurcations of system (3.2). This study will use some results obtained in [13]: in particular, the results on the cyclicity of a singular point of (3.2). However, it is surely not enough to have only these results to prove the main theorem of this paper concerning the maximum number of limit cycles of system (3.2).

Finally, we will see also that the main result of this paper is quite similar to the main result of [3], where a Holling system of type IV was studied, but the number of singular points in the first quadrant and the distribution of limit cycles in the two systems are different.

Applying the definition of a field rotation parameter [1, 4, 14], i. e., a parameter which rotates the field in one direction, to system (3.2), let us calculate now the corresponding determinants for the parameters $\alpha$ and $\beta$, respectively:

$$\Delta_\alpha = PQ'_\alpha - QP'_\alpha = x^4 y(y(\delta + \mu y) - x(1 - \lambda x)), \tag{3.12}$$

$$\Delta_\beta = PQ'_\beta - QP'_\beta = x^4 y(y(\delta + \mu y) - x(1 - \lambda x)). \tag{3.13}$$

It follows from (3.12) and (3.13) that on increasing $\alpha$ or $\beta$ the vector field of (3.2) in the first quadrant is rotated in the positive direction (counterclockwise) only on the outside of the ellipse

$$y(\delta + \mu y) - x(1 - \lambda x) = 0. \tag{3.14}$$

Therefore, to study limit cycle bifurcations of system (3.2), it makes sense together with (3.2) to consider also an auxiliary system (3.3) with a field rotation parameter $\gamma$:

$$\Delta_\gamma = P^2 + Q^2 \geq 0. \tag{3.15}$$

Using system (3.3) and applying Perko's results [4, 14], we prove the following theorem.

**Theorem 3.3.** *System* (3.2) *can have at most two limit cycles surrounding one singular point.*

*Proof.* First let us prove that system (3.2) can have at least two limit cycles. Begin with system (3.2), where $\alpha = \beta = 0$. It is clear that such a cubic system, with two invariant straight lines, cannot have limit cycles at all [13]. Inputting a negative parameter $\beta$ into this system, the vector field of (3.2) will be rotated in the negative direction (clockwise) at infinity, the structure and the character of stability of infinite singularities will be changed, and an unstable limit, $\Gamma_1$, will appear immediately from infinity in this case. This cycle will surround a stable anti-saddle (a node or a focus) $A$ which is in the first quadrant of system (3.2). Inputting a positive parameter $\alpha$, the vector field of quartic system (1.10) will be rotated in the positive direction (counterclockwise) at infinity, the structure and the character of stability of infinite singularities will be changed again, and a stable limit, $\Gamma_2$, surrounding $\Gamma_1$ will appear immediately from infinity in this case. On further increasing the parameter $\alpha$, the limit cycles $\Gamma_1$ and $\Gamma_2$ combine a semi-stable limit, $\Gamma_{12}$, which then disappears in a "trajectory concentration" [1, 4]. Thus, we have proved that system (3.2) can have at least two limit cycles; see also [13].

Let us prove now that this system has at most two limit cycles. The proof is carried out by contradiction applying catastrophe theory [4, 14]. Consider system (3.3) with three parameters: $\alpha$, $\beta$, and $\gamma$ (the parameters $\delta$, $\lambda$, and $\mu$ can be fixed, since they do not generate limit cycles). Suppose that (3.3) has three limit cycles surrounding the only point $A$ in the first quadrant. Then we get into some domain of the parameters $\alpha$, $\beta$, and $\gamma$ being restricted by definite conditions on three other parameters $\delta$, $\lambda$, and $\mu$. This domain is bounded by two fold bifurcation surfaces forming a cusp bifurcation surface of multiplicity-three limit cycles in the space of the parameters $\alpha$, $\beta$, and $\gamma$.

The corresponding maximal one-parameter family of multiplicity-three limit cycles cannot be cyclic, otherwise there will be at least one point corresponding to the limit cycle of multiplicity four (or even higher) in the parameter space.

Extending the bifurcation curve of multiplicity-four limit cycles through this point and parameterizing the corresponding maximal one-parameter family of multiplicity-four limit cycles by the field rotation parameter $\gamma$, according to Theorem 3.2, we will obtain two monotonic curves of multiplicity-three and one, respectively, which, by the Wintner–Perko termination principle (Theorem 3.1), terminate either at the point $A$ or on a separatrix cycle surrounding this point.

Since we know at least the cyclicity of the singular point which is equal to two [13], we have got a contradiction with the termination principle stating that the multiplicity of limit cycles cannot be higher than the multiplicity (cyclicity) of the singular point in which they terminate.

If the maximal one-parameter family of multiplicity-three limit cycles is not cyclic, using the same principle (Theorem 3.1), this again contradicts the cyclicity of $A$ [13] not admitting the multiplicity of limit cycles to be higher than two. This contradiction completes the proof in the case of one singular point in the first quadrant.

Suppose that system (3.3) with two finite singularities, a saddle $S$ and an anti-saddle $A$, has three limit cycles surrounding $A$. Then we get again into some domain of the parameters $\alpha$, $\beta$, and $\gamma$ bounded by two fold bifurcation surfaces forming a cusp bifurcation surface of multiplicity-three limit cycles in the space of the parameters $\alpha$, $\beta$, and $\gamma$ being restricted by definite conditions on three other parameters $\delta$, $\lambda$, and $\mu$.

The corresponding maximal one-parameter family of multiplicity-three limit cycles cannot be cyclic, otherwise there will be at least one point corresponding to the limit cycle of multiplicity four (or even higher) in the parameter space. Extending the bifurcation curve of multiplicity-four limit cycles through this point and parameterizing the corresponding maximal one-parameter family of multiplicity-four limit cycles by the field rotation parameter $\gamma$, according to Theorem 3.2, we will obtain again two monotonic curves of multiplicity-three and one, respectively, which, by Theorem 3.1, terminate either at the point $A$ or on a separatrix loop surrounding this point [4].

Since we know at least the cyclicity of the singular point which is equal to two [13], we have got a contradiction with the termination principle (Theorem 3.1).

If the maximal one-parameter family of multiplicity-three limit cycles is not cyclic, using the same principle, this again contradicts the cyclicity of $A$ [13] not admitting the multiplicity of limit cycles higher than two. Moreover, it also follows from the termination principle that a separatrix loop cannot have the multiplicity (cyclicity) higher than two in this case.

Thus, we conclude that system (3.2) cannot have either a multiplicity-three limit cycle or more than two limit cycles surrounding a singular point which proves the theorem. □

## References


1. *Bautin, N. N., Leontovich, E. A.* Methods and Examples of the Qualitative Analysis of Dynamical Systems in a Plane. — Moscow: Nauka, 1990. — 488 p. (in Russian).
2. *Botelho, F., Gaiko, V. A.* Global analysis of planar neural networks // Nonlinear Anal. — 2006. — Vol. 64, No. 1. — P. 1002–1011.
3. *Broer, H. W., Gaiko, V. A.* Global qualitative analysis of a quartic ecological model // Nonlinear Anal. — 2010. — Vol. 72, No. 2. — P. 628–634.
4. *Gaiko, V. A.* Global Bifurcation Theory and Hilbert's Sixteenth Problem. — Boston: Kluwer Academic Publishers, 2003. — 204 p.
5. *Gaiko, V. A., van Horssen, W. T.* Global bifurcations of limit and separatrix cycles in a generalized Liénard system // Nonlinear Anal. — 2004. — Vol. 59, No. 1–2. — P. 189–198.
6. *Gaiko, V. A.* Limit cycles of quadratic systems // Nonlinear Anal. — 2008. — Vol. 69, No. 7. — P. 2150–2157.
7. *Gaiko, V. A.* Multiple limit cycle bifurcations of the FitzHugh–Nagumo neuronal model // Nonlinear Anal. — 2011. — Vol. 74, No. 18. — P. 7532–7542.
8. *Gaiko, V. A.* On limit cycles surrounding a singular point // Differ. Equ. Dyn. Syst. — 2012. — Vol. 20, No. 3. — P. 329–337.

9. *Gaiko, V. A.* The applied geometry of a general Liénard polynomial system // Appl. Math. Letters. — 2012. — Vol. 25, No. 12. — P. 2327–2331.
10. *Gaiko, V. A.* Limit cycle bifurcations of a general Liénard system with polynomial restoring and damping functions // Int. J. Dyn. Syst. Differ. Equ. — 2012. — Vol. 4, No. 3. — P. 242–254.
11. *Gaiko, V. A., Ginoux, J.-M., Vuik, C.* Control of limit cycle bifurcations in the Kukles cubic system // Eighth International Conference on Physics and Control [Electronic resource]: Proc. / A. L. Fradkov [et al.], Eds. — Florence, 2017. — 6 p. — http://coms.physcon.ru/getpaper/c1608p25r4164.pdf/, 17.07.2017.
12. *Gaiko, V. A., Ginoux, J.-M., Vuik, C.* The termination principle of multiple limit cycles for the Kukles cubic system // Cyber. Phys. — 2017. — Vol. 6, No. 4. — P. 195–200.
13. *Lamontagne, Y., Coutu, C., Rousseau, C.* Bifurcation analysis of a predator-prey system with generalized Holling type III functional response // J. Dyn. Diff. Equations. — 2008. — Vol. 20. — P. 535–571.
14. *Perko, L.* Differential Equations and Dynamical Systems. — New York: Springer–Verlag, 2002. — 555 p.
15. *Smale, S.* Mathematical problems for the next century // Math. Intellig. — 1998. — Vol. 20. — P. 7–15.
16. *Vuik, C., Vermolen, F. J., van Gijzen, M. B., Vuik, M. J.* Numerical Methods for Ordinary Differential Equations. — Delft: Delft Academic Press, 2015. — 212 p.

*Получена 12.02.2018*