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# Topological conjugacy of gradient-like flows on surfaces[1]

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**Abstract.** The class of $C^1$-smooth gradient-like flows (Morse flows) on closed surface is the subclass of the Morse-Smale flows class, which are rough. Their non-wandering set consists of a finite number of hyperbolic fixed points and a finite number of hyperbolic limit cycles, and they does not have trajectories connecting saddle points. It is well known that the topological equivalence class of a Morse-Smale flow on a surface can be described combinatorially, for example, by the directed Peixoto graph, or by the Oshemkov-Sharko molecule. However, the description of the class of the topological conjugacy of such a system already requires the introduction of continuous invariants (moduli), corresponding to the periods of limit cycles at least. Thus, one class of the equivalence contains continuum classes of the topological conjugacy. Gradient-like flows are Morse-Smale flows without limit cycles. In this paper we prove that gradient-like flows on a closed surface are topologically conjugate iff they are topologically equivalent.
**Keywords:** gradient-like flow, Morse-Smale flow, conjugacy, equivalence, homeomorphism

## 1. Introduction and formulation of results

In 1937 A. Andronov and L. Pontryagin published the classical paper [1], in which they considered a system of differential equations

$$\dot{x} = v(x), \tag{1.1}$$

where $v(x)$ is a $C^1$-vector field given on a disc bounded by a curve without a contact in the plane and found a roughness criterion for the system (1.1). They established that on the plane the rough system is exactly system whose non-wandering set consists of finite number of hyperbolic fixed points and hyperbolic limit cycles and which does not have trajectories connecting saddle points. Such systems were called *Morse-Smale systems* when in 1967 S. Smale generalised such systems to multidimensional case in [8]. If a Morse-Smale system does not have limit cycles, then it is called as *Morse system* or *gradient-like system.*

The present paper is devoted to the classification of Morse flows on a closed surfaces $S$.

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Let us recall that two flows $f^t$ and $f'^t$ on surface $S$ are called *topologically equivalent* if there exists a homeomorphism $h\colon S \to S$ mapping trajectories of one flow into trajectories of another preserving directions of moving. Flows $f^t$ and $f'^t$ on surface $S$ are called *topologically conjugate* if there exists a homeomorphism $h\colon S \to S$ such that $h \circ f^t = f'^t \circ h$ for every real $t$.

It is well known that the topological equivalence class of the Morse-Smale flow on surface can be described combinatorially, for example, by the directed Peixoto graph, or by the Oshemkov-Sharko molecule. In more details.

The directed Peixoto graph introduced by him in 1971 in [6] for arbitrary Morse-Smale flow on a closed surface, is the generalisation of the Leontovich-Mayer scheme, introduced in [2] (1937) and [3] (1955) for flows on the plane (but not only Morse-Smale). Their approach is based on the ideas of Poincaré-Bendixon to pick a set of specially chosen trajectories so that their relative position fully determines the qualitative decomposition of the phase space of the flow into the trajectories. The Peixoto graph's vertices bijectively correspond to fixed points and limit cycles of the flow, its edges correspond to the connected components of the invariant manifolds of fixed points and closed trajectories without the points and the trajectories itself (see Fig. 1). To be a complete topological invariant such graphs contain the specially chosen subgraphs as well.

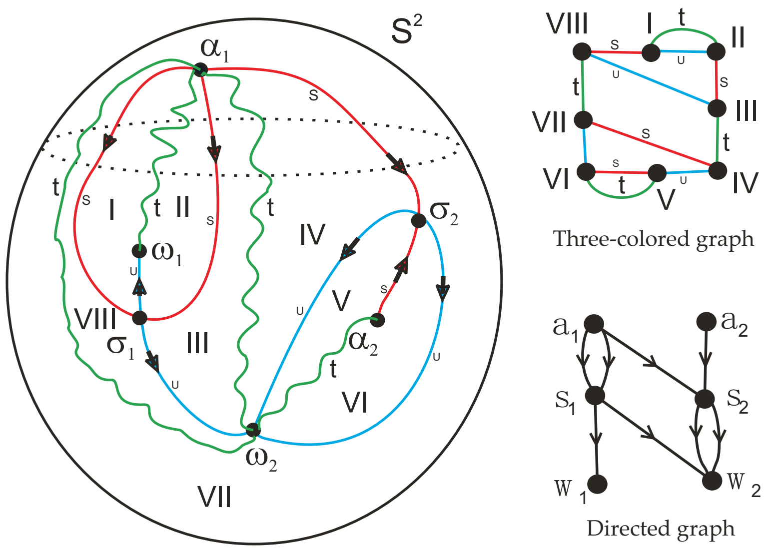


Fig. 1. An example of a gradient-like flow on a sphere $S^2$, its Peixoto's directed graph and its Oshemkov-Sharko's three-colour graph

However, in 1998 Oshemkov and Sharko in [4] found that the Peixoto's graph is not complete for all Morse-Smale flows, especially it does not always distinguish the difference between two types of decompositions into trajectories for a domain bounded by two limit cycles of the flow. For Morse flows they introduced the new complete invariant – *three-colour graph*, its vertices correspond to the so-called triangular domains, restricted by two saddle separatrices and one usual trajectory which are called *sides*, and these sides correspond to coloured edges of the graph, side of each type corresponds to the edge of the certain colour. Then Oshemkov and Sharko in the same work took three-colour graphs and elementary domains with simple behaviour as atoms and constructed with these atoms *the molecules*, and proved that such molecules are surely complete topological invariant for Morse-Smale flows on surfaces.

A description of the class of the topological conjugacy of Morse-Smale flows, in a difference with the equivalence, requires an introduction of continuous invariants (moduli), corresponding with the periods of the limit cycles at least. Thus, one class of the equivalence contains continuum classes of the topological conjugacy. In this paper we show that for gradient-like systems these classes are coincide, namely we prove the following fact.

**Theorem 1.** *If two gradient-like flows on a closed surface are topological equivalent then they are topologically conjugate.*

## 2. Necessary facts and statements

**Definition 1.** A map $h$ of a metric space $(X, \rho_X)$ to a metric space $(Y, \rho_Y)$ is called *Lipschitz*, if there is some positive constant $L$ called as *Lipschitz constant* such that $\rho_Y(h(x), h(y)) \leq L \cdot \rho_X(x, y)$ for all $x, y \in X$.

Let $\tilde{C}^0(\mathbb{R}^n)$ be the Banach space of bounded continuous maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ with uniform norm $||u|| = \sup\{||u(x)|| : x \in \mathbb{R}^n\}$.

**Proposition 1** ([5], Ch. 2, Lemma 4.3)**.** Let $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ be a hyperbolic isomorphism. Then there exists a value $\varepsilon > 0$ such that for every $\varphi_1, \varphi_2 \in \tilde{C}^0(\mathbb{R}^n)$ with the Lipschitz constant less or equal than $\varepsilon$ there is an unique continuous map $h : \mathbb{R}^n \to \mathbb{R}^n$ of the form

$$h = I + u,$$

where $I$ is the identity map and $u \in \tilde{C}^0(\mathbb{R}^n)$, such that

$$h(\phi + \varphi_1) = (\phi + \varphi_2)h.$$

Moreover $h$ is a homeomorphism.

**Proposition 2** ([5], Ch. 2, Lemma 4.9)**.** Let $F\colon \mathbb{R}^n \to \mathbb{R}^n$ be a $C^r$-vector field with the equilibrium point 0. Then for every $\varepsilon > 0$ there exists a $C^r$-vector field $G\colon \mathbb{R}^n \to \mathbb{R}^n$ and neighborhoods $U \subset V$ of 0 such that:

1) $G = F$ on $U$ and $G = DF_0$ outside $V$;

2) $G$ is Lipschitz and generates a flow $g^t$ on $\mathbb{R}^n$ of the form

$$g^t = \phi^t + \varphi^t,$$

where $\phi^t$ is a flow generated by the vector field $DF_0$, $\varphi^t \in \tilde{C}^0(\mathbb{R}^n)$ for all $t \in [-2, 2]$, $\varphi^1$ has the Lipschitz constant less than $\varepsilon$ and $D\varphi^1_0 = 0$.

Now let $M^n$ be a $C^r$-smooth $n$-manifold.

**Lemma 1.** *Let $F\colon M^n \to M^n$ be a $C^r$-vector field with the hyperbolic equilibrium point $p$. Then there exists a neighbourhood $U$ of the point $p$ where the flow $f^t$ generated by $F$ is topologically conjugated to the flow $\phi^t$ generated by $DF_p$.*

*Proof.* As the problem is local, and there exists some local map $(V, \theta)$, where $U \subset V$, $\theta\colon V \to \mathbb{R}^n$ is homeomorphism and $\theta(p) = 0$, let us think that $M^n = \mathbb{R}^n$ and $p = 0$.

As 0 is the hyperbolic equilibrium point of $F$ then $\phi = \phi^1$ is the hyperbolic isomorphism of $\mathbb{R}^n$. Let $\varepsilon$ be a constant from Proposition 1 for $\phi$ and $G$ be the vector field from Proposition 2 for $F$ and $\varepsilon$. Then the flows $f^t$ and $g^t$ generated by $F$ and $G$, accordingly, are coincide on $U$ and, hence, they are topologically conjugate on $U$. Using an idea of the proof of Theorem 4.10 from Ch. 2 of [5], let us show that the flow $\phi^t$ is topologically conjugate to $g^t$ in $\mathbb{R}^n$.

By Propositions 1 and 2 there exists an unique homeomorphism $h\colon \mathbb{R}^n \to \mathbb{R}^n$ being in a finite distance from the identity map such that $hg = \phi h$. Let

$$H = \int\limits_0^1 \phi^{-t} h g^t dt.$$

This map is continuous and, by Proposition 2, is in a finite distance from the identity map. Let us show that $\phi^s H = H g^s$ for all $s \in \mathbb{R}$, all we need for this is to consider the segment from 0 to 1, because it is fundamental. Let us take and fix some $s$ from $[0, 1]$. We have

$$\phi^{-s} H g^s = \phi^{-s} \left( \int\limits_0^1 \phi^{-t} h g^t dt \right) g^s = \int\limits_0^1 \phi^{-(t+s)} h g^{t+s} dt.$$

Let $u = t + s - 1$, then

$$\int\limits_0^1 \phi^{-(t+s)} h g^{t+s} dt = \int\limits_{-1+s}^{s} \phi^{-u-1} h g^{u+1} du =$$

$$= \int\limits_{-1+s}^{0} \phi^{-u} \phi^{-1} h g^1 g^u du + \int\limits_0^s \phi^{-u-1} h g^{u+1} du.$$

Let $v = u+1$ in the first sum and $v = u$ in the second one and recall that $\phi^{-1} h g^1 = h$. It gives us the formula

$$\phi^{-s} H g^s = \int\limits_0^s \phi^{-v} h g^v du + \int\limits_s^1 \phi^{-v} h g^v du = H.$$

It implies that $H$ is the continuous map being in a finite distance from the identity map and conjugating the flow $\phi^t$ with $g^t$. As $hg^1 = \phi^1 h$ and $Hg^1 = \phi^1 H$, uniqueness of solving of this equation gives $h = H$. □

**Proposition 3** ([7], Ch. 4, Theorem 7.1)**.** Let $A$ and $B$ be two $n \times n$ real matrices such that all the eigenvalues of $A$ and $B$ have nonzero real part and the dimension of the direct sum of all the eigenspaces with negative (and, obviously, positive too) real part is the same for $A$ and $B$. Then the two flows generated by the vector fields $\dot{x} = Ax$ and $\dot{x} = Bx$ are topologically conjugate.

## 3. The proof of the main theorem

Let $S$ be a closed surface and $f^t\colon S \times \mathbb{R} \to S$ be a $C^1$ gradient-like flow. Then for every wandering trajectory $\ell$ of $f^t$ there are exactly two different fixed points $p, q$ of $f^t$ such that the boundary of the trajectory has the form

$$\mathrm{cl}(\ell)\backslash\ell = \{p, q\}$$

and the trajectory is directed from $p$ to $q$. In this case we will denote the trajectory by $\ell_{p,q}$ assuming that the trajectory is directed from $p$ to $q$.

Let $f^t$ and $f'^t$ be topologically equivalent $C^1$ gradient-like flows, i.e. there is a homeomorphism $h\colon S \to S$ mapping trajectories of $f^t$ into trajectories of $f'^t$ preserving orientation. It implies that $h$ maps the fixed points of $f^t$ to the fixed points of $f'^t$, what we will denote by $p' = h(p)$ for a fixed point $p$ of $f^t$. Then

$$h(\ell_{pq}) = \ell'_{p'q'}$$

for every wandering trajectory $\ell_{pq}$ of $f^t$.

By Lemma 1 and Proposition 3 there are neighbourhoods $u_p$, $u_{p'}$ of $p$, $p'$ respectively such that $f^t|_{u_p}$, $f'^t|_{u_{p'}}$ are topologically conjugated by a homeomorphism $h_p : u_p \to u_{p'}$.

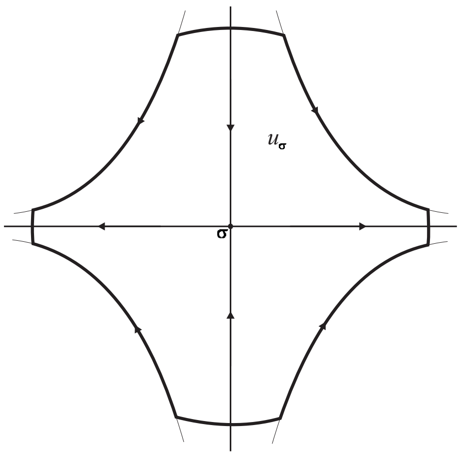


Fig. 2. Neighbourhood $u_\sigma$

Let $\sigma$ be a saddle point of $f$. Without loss of generality we will assume that the neighborhood $u_\sigma$ has a form as on Figure 2, $u_{\sigma'} = h_\sigma(u_\sigma)$ and a map $h^{-1}h_\sigma$ preserves separatrix of $\sigma$. For a point $x \in S$ denote by $\mathcal{O}_x$ ($\mathcal{O}'_x$) the orbit of the flow $f^t$ ($f'^t$) passing through the point $x$. Let

$$V_\sigma = \bigcup_{x \in \mathrm{cl}(u_\sigma)} \mathcal{O}_x, \quad V_{\sigma'} = \bigcup_{x \in \mathrm{cl}(u_{\sigma'})} \mathcal{O}'_x.$$

Let us extend $h_\sigma$ up to a homeomorphism $h_{V_\sigma} \colon V_\sigma \to V_{\sigma'}$ by the following rule (see Fig. 3). For a point $z \in (V_\sigma \backslash \mathrm{cl}(u_\sigma))$ let $\{z_0\} = \mathcal{O}_z \cap \partial u_\sigma$ and $f^{t_z}(z_0) = z$ for $t_z \in \mathbb{R}$, then

$$h_{V_\sigma}(z) = f'^{t_z}(h_\sigma(z_0)).$$

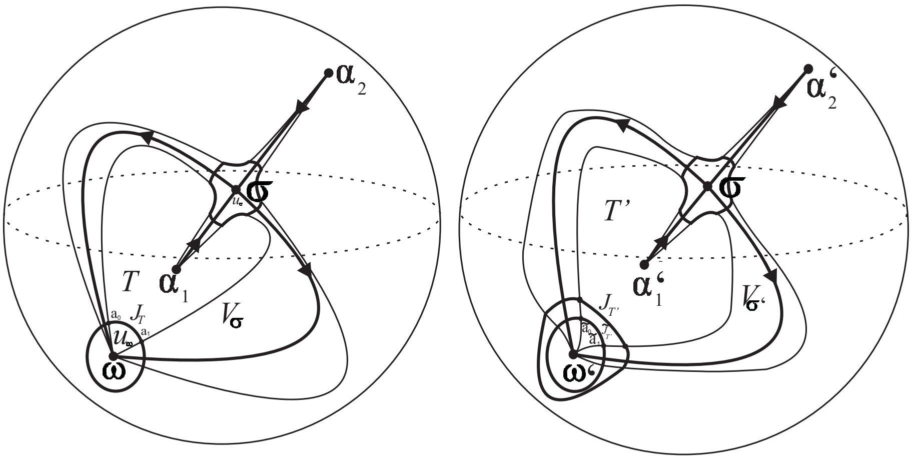


Fig. 3. Main constructions for $f^t$ (on the left sphere) and for $f'^t$ (on the right sphere)

Let $V$ ($V'$) be a union of all $V_\sigma$ ($V_{\sigma'}$) and $h_V : V \to V'$ be a homeomorphism composed by $h_{V_\sigma}$.

To extend the homeomorphisms $h_V$ up to ambient conjugating homeomorphism note that the closure $T$ of any connected component of the set $S \setminus (V \cup \Omega_{f^t})$ belongs to the basin of a sink $\omega$. As $h^{-1}h_\sigma$ preserves separatrix of $\sigma$ then there is the closure $T' \subset W^s_{\omega'}$ of an unique connected component of the set $S \setminus (V' \cup \Omega_{f'^t})$ such that $h(T) \cap T' \neq \emptyset$. Let us extend $h_V$ to $T$ by conjugating homeomorphism $h_T$.

By Lemma 1 flows $f^t|_{u_\omega}$ and $f'^t|_{u_{h(\omega)}}$ are conjugate by means of $\psi_\omega$ and $\psi_{\omega'}$ respectively to some linear flows in some neighbourhood of 0 on the plane. Let $\gamma_0$ be some closed curve without a contact, transversally crossing all trajectories of the linear flows, and let $\gamma = \psi_\omega^{-1}(\gamma_0)$, $\gamma' = \psi_{\omega'}^{-1}(\gamma_0)$. So we correctly constructed a closed curve without a contact around $\omega$ and $\omega'$.

Let $J_T = \gamma \cap T$ and let $a_0, a_1$ be the endpoints of the arc $J_T$. Then there are saddle points $\sigma_0, \sigma_1$ (possible $\sigma_0 = \sigma_1$) such that $a_i \in (J_T \cap V_{\sigma_i}), i = 0, 1$. Similarly the arc $\tilde{J}_{T'} = \gamma' \cap T'$ is bounded by the points $\tilde{a}_0, \tilde{a}_1$ belonging to $V_{\sigma'_0}, V_{\sigma'_1}$, accordingly. Let $t_0, t_1 \in \mathbb{R}$ so that $f'^{t_i}(\tilde{a}_i) = h_V(a_i), i = 0, 1$ and $\rho \colon \tilde{J}_{T'} \to [0, 1]$ be a homeomorphism such that $\rho(\tilde{a}_i) = i, i = 0, 1$. Let

$$J_{T'} = \{f'^{t_z}(\tilde{z}) \mid \tilde{z} \in \tilde{J}_{T'}, t_z = t_0 + (t_1 - t_0)\rho(\tilde{z})\}.$$

Define an arbitrary homeomorphism $h_J\colon J_T \to J_{T'}$ so that $h_J(a_i) = h_V(a_i), i = 0, 1$. Then every point $z$ in $T$ is uniquely defined by the point $z_0 = \mathcal{O}_z \cap J_T$ and the value $t_z \in \mathbb{R}$ such that $f^{t_z}(z_0) = z$. Let us define a homeomorphism $h_T\colon T \to T'$ by the formula

$$h_T(f^{t_z}(z_0)) = f'^{t_z}(h_J(z_0)).$$

Let us define the conjugating homeomorphism $h_c\colon S \to S$ so that $h_c|_V = h_V$, $h_c|_T = h_T$ and $h_c|_{\Omega_{f^t}} = h|_{\Omega_{f^t}}$. Thus the conjugating homeomorphism is constructed and Theorem is proved.

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