Динамические системы, 2017, том 7(35), №2, 131–148

MSC 2010: 74P05, 74C15, 65H05, 90C05

Geometrically and physically nonlinear optimization problem for the 3-bar truss

V. V. Chekhov

V. I. Vernadsky Crimean Federal University,
4 Vernadskogo Prospekt, Simferopol, 295007 *E-mail: v_chekhov@ukr.net*

Abstract. The analysis and optimal design problems are considered for the statically loaded symmetric 3-bar truss with taking into account material nonlinearity and finite deformation. The strain compatibility condition as well as the relationship between the cross-sectional areas of bars and the stress levels in them was derived for the finite deformation. It is shown that even though the problem of analysis of the truss is a nonlinear one, the optimization problem is a linear programming problem. Properties of optimal designs are compared using various structural alloys.

Keywords: symmetric 3-bar truss, physical nonlinearity, geometrical nonlinearity, optimal design problem.

1. Introduction

The use of simple test models is of great benefit for testing the methods of structural analysis/optimization. Such models allow you to check results of the used algorithm by using known solutions and also to represent a complicated process of loading or designing the structure in an obvious form which is suitable for the aid of comprehension or training. Among such simple models, a symmetrical flat three-bar truss is often found. This is one of the simplest static indeterminate structures. There is a rather detailed estimation of the stress-strain state and the safety margin of the truss in educational literature, both in the linear-elastic formulation and the one with taking into account plastic properties of the materials used (physical nonlinearity). For example, the solution of this task in [3] is given under symmetrical loading for the truss with an arbitrarily defined angle between the rods using different materials in the central and lateral rods. The analysis of the truss deformation with the occurrence of finite deformation (geometric nonlinearity) represents a more difficult task. Currently, it is used for test calculations in works devoted to methods of geometrically nonlinear structural analysis [9, 10, 13].

When considering the optimal design problem for the 3-bar truss, values of the cross-sectional area of the rods are usually assumed as the design parameters. The problem includes constraints on the stress levels in the rods [1], as well as constraints from below on the design parameters: zero (to correspond the physical meaning of the problem) or or positive (known as the tecnology constraints) to prevent degeneration of the rods. Other types of constraints can also be added to the task [7]: the displacement value in the loaded node, stability and/or the natural frequency. The above constraints can be specified simultaneously for several different loading cases, which contain different values and directions of the external load. Usually, two loading cases are considered [1, 4, 7]. In these publications, the truss is considered with the corner angle $\pi/4$ and with the behavior based on the geometrically and physically linear-elastic finite element model. When optimizing the truss by taking into account the geometrical nonlinearity, the consideration is most often limited to one loading case with a symmetrically applied tensile external load [9, 11, 13]. The design of the single-material truss is considered in the mentioned publications.

If the ability to specify its own material for each type of bar is added to the problem, then an additional degree of freedom appears, resulting in phenomena that are not present using just one material. In particular, it is possible here that the optimal design obtained with allowance for physical nonlinearity proves to be heavier than the same linearelastic one [12]. This contradicts the widespread view that the use of the linear-elastic model for materials leads to the design "in reserve" and the growth of the structural mass.

When taken into account, the material plasticity results in the increased level of displacement and strain in comparison with the linear-elastic case. Therefore it is necessary to take into account the geometrical nonlinearity to ensure the reliability of results. Theoretically, this is difficult to do and, therefore, large deformations are currently taken into account, usually only in practical calculations, using modern packages of the finite element method. However, the simplicity of a 3-bar truss makes it possible to theoretically consider the problem of optimal design taking into account possible finite deformations and to estimate their influence on the accuracy of the calculations. The results of such a study are presented in this paper.

2. Description of the analitical model

A symmetric three-bar truss shown in Fig. 1 is considered. The truss is symmetrically statically loaded at the point of intersection of the rods by an external constant tensile force P. The possible appearance of large deformation under the load is taken into account, so the initial state and the deformed one are assumed to be different (for comparison, the similar relations for small deformation are given). All quantities related to the initial configuration are denoted by a zero in the upper index. The angle α^0 between the rods can be specified arbitrarily in the range $0 \leqslant \alpha^0 < \pi/2$. All values related to the central bar are denoted by the subscript "c", and the quantities related to the lateral bars have the subscript "s". Two materials which may be different are set in the central bar and in the lateral ones. We confine ourselves to the consideration of standard structural alloys whose physically nonlinear behavior can be described using the deformation plasticity theory [3]. The notation of the quantities and the scheme of the deformation of the truss are shown in Fig. 2. It is obvious that only one kind of rods (central or lateral) can be deformed arbitrarily, which uniquely determines the deformation of the other rod(-s). Let's define the compatibility condition for deformations for the general geometrically nonlinear case.

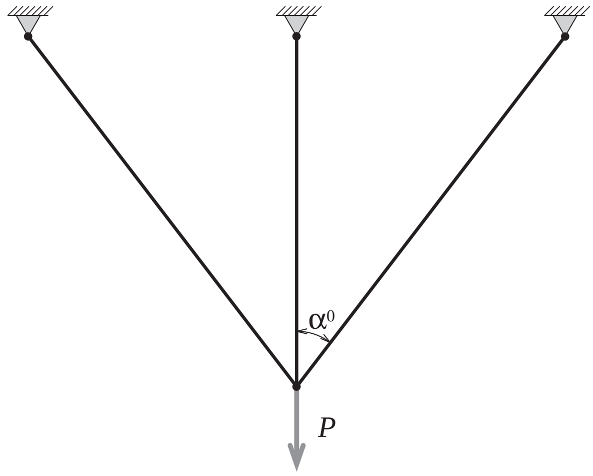

Fig. 1. The 3-bar truss

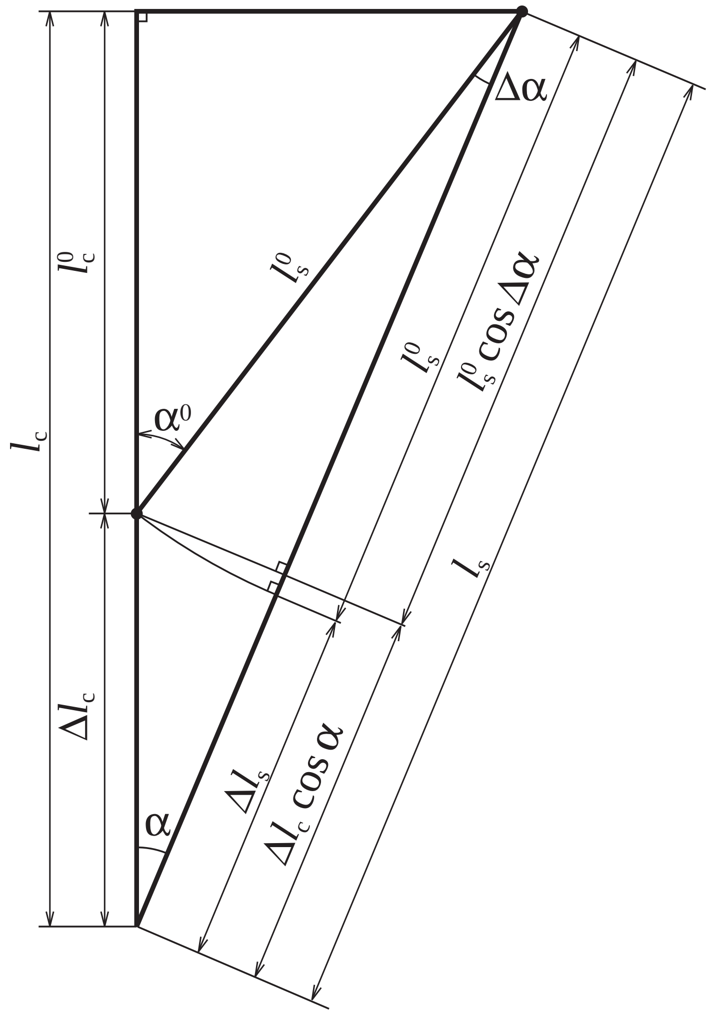

Fig. 2. The deformation of the truss

3. The compatibility condition under an arbitrary deformation

Using the notation of figure 2, we can write the following obvious relationships:

$$l_c^0 = l_s^0 \cos\alpha^0, \tag{3.1}$$

$$l_c = l_s \cos\alpha, \tag{3.2}$$

$$l_c = l_c^0 + \Delta l_c, \tag{3.3}$$

$$\alpha^0 = \alpha + \Delta\alpha, \tag{3.4}$$

$$l_s = l_s^0 \cos\Delta\alpha + \Delta l_c \cos\alpha. \tag{3.5}$$

By succesive elimination of the following terms from relation (3.5): Δl_c with the aid of (3.3), l_c by the use of (3.2), $\Delta\alpha$ by virtue of (3.4), and l_c^0 by means of (3.1), while carrying out the necessary transformations and reductions, we obtain a fairly compact expression

$$\frac{l_s}{l_s^0} = \frac{\sin\alpha^0}{\sin\alpha}.$$

Similarly, eliminating the following terms from (3.5) successively: Δl_c with the aid of (3.3), l_s by the use of (3.2), $\Delta\alpha$ by virtue of (3.4), and l_s^0 by means of (3.1), we derive an expression

$$l_c = l_c^0 \frac{\operatorname{tg}\alpha^0}{\operatorname{tg}\alpha}.$$

These conditions can be more conveniently expressed in terms of stretch

$$\lambda = \frac{l}{l^0}$$

which is an important value in the theory of large deformations.

$$\lambda_s = \frac{\sin\alpha^0}{\sin\alpha}, \qquad \lambda_c = \frac{\operatorname{tg}\alpha^0}{\operatorname{tg}\alpha}, \qquad \frac{\lambda_s}{\lambda_c} = \frac{\cos\alpha^0}{\cos\alpha}. \tag{3.6}$$

After squaring the last expression and eliminating $\sin\alpha$ from it using the first expression, we derive after cancellations the strain compatibility condition for the 3-bar truss:

$$\lambda_s^2 = \lambda_c^2 \cos^2\alpha^0 + \sin^2\alpha^0. \tag{3.7}$$

If we express the compatibility condition in terms of strain

$$\varepsilon = \lambda - 1, \tag{3.8}$$

then the first and second expressions (3.6) take the form

$$\varepsilon_s = \frac{\sin\alpha^0 - \sin\alpha}{\sin\alpha}, \qquad \varepsilon_c = \frac{\operatorname{tg}\alpha^0 \cos\alpha - \sin\alpha}{\sin\alpha}.$$

Dividing the first expression by the second one, after some transformations, we get

$$\frac{\varepsilon_s}{\varepsilon_c} = \frac{\sin\alpha^0 \sin\Delta\alpha}{1+\cos\Delta\alpha} + \cos^2\alpha^0.$$

At small deformation $\Delta\alpha \to 0$, whence $\sin\Delta\alpha \to 0$, $\cos\Delta\alpha \to 1$. As a result, we obtain the well-known formulation of the compatibility condition for small deformations:

$$\frac{\varepsilon_s}{\varepsilon_c} = \cos^2\alpha^0. \tag{3.9}$$

In particular, $\alpha^0 = \pi/4$ gives $\varepsilon_c = 2\varepsilon_s$.

4. Relationship between stresses in bars and their cross-sectional areas

In projection to the axis of the central rod, the equilibrium equation for the forces at the acting load point has the following form:

$$N_c + 2N_s \cos\alpha = P.$$

Here $N_c = \sigma_c F_c$, $N_s = \sigma_s F_s$ — internal forces at the rods, σ_c, σ_s — stresses at the rods, F_c, F_s — the cross-sectiona areas of the rods. Eliminating $\cos\alpha$ from this equation by using (3.6), we express the internal forces in terms of the stresses:

$$\sigma_c F_c + 2\sigma_s F_s \frac{\lambda_c}{\lambda_s}\cos\alpha^0 = P.$$

In addition, it is possible to take into account the true stresses in this equation. Allowing for the fact that the cross-sectional area of a bar decreases while stretching the bar, according to the generalized Hooke's law [3] (here it is desirable for the materials to specify the true stress-strain diagrams), the equation takes on form:

$$\sigma_c(1-\nu_c\varepsilon_c)^2 F_c^0 + 2\sigma_s\frac{\lambda_c}{\lambda_s}(1-\nu_s\varepsilon_s)^2 F_s^0 \cos\alpha^0 = P \tag{4.1}$$

where ν_c, ν_s are the Poisson ratios of the used materials. Thus, the equation has been obtained linking the cross-sectional areas of the rods with the stresses in them.

Equation (4.1) connects four parameters (the cross-sectional areas and the stresses). Now, if we join it with the compatibility condition (3.7) (which connects the stresses — two of these parameters) then we obtain a system which allows to specify values for the two parameters and to derive values for the other two. The most simple from the available possibilities is to set the stress in one of the rods and the cross-sectional area of one of the rods (the choice of the rod is arbitrary in both cases). The procedure is the following:

1. the strain value is obtained from the specified stress by the use of the stress-strain curve of the bar;

2. the stretch value is obtained from the strain value by means of (3.8);
3. the stretch of the other bar is obtained by the compatibility condition (3.7) from the already known stretch;
4. the strain of the other bar is obtained from the its stretch;
5. the stress in the other bar is obtained from the its strain;
6. already known values of stress, strain and stretch are substituted in equation (4.1);
7. the specified value of the cross-sectional area of one of the bars is substituted in this equation;
8. the derived equality is the linear equation from which the cross-sectional area of the residual bar is expressed.

It should be noted that even when physical and geometric nonlinearities are taken into account, equation (4.1) *linearly* connects the cross-sectional areas F_c^0, F_s^0 of the rods. Therefore, in the plane of the variables F_c^0, F_s^0, lines of equal stresses will always be the straight ones intersecting the coordinate axes at the points

$$\{F_c^0 = 0, \quad F_s^0 = \frac{\lambda_s}{2\lambda_c}\frac{P}{\sigma_s(1-\nu\varepsilon_s)^2\cos\alpha^0} = \frac{P}{\sigma_s(1-\nu\varepsilon_s)^2\cos\alpha}\},$$

$$\{F_c^0 = \frac{P}{\sigma_c(1-\nu\varepsilon_c)^2}, \quad F_s^0 = 0\}$$

Similarly, in the case of small deformations, according to (4.3), the intersection points of the equal stress lines with the coordinate axes are

$$\{F_c^0 = 0, \quad F_s^0 = \frac{P}{2\sigma_s\cos\alpha^0}\}, \qquad \{F_c^0 = \frac{P}{\sigma_c}, \quad F_s^0 = 0\}.$$

Using the system (4.1), (3.7), we can also solve the direct problem of calculating the stress values in rods for given areas of their cross-section. To do this, from equation (3.7) we express the value of one of the stretches through the other one (noting that for the problem being solved the physical meaning exists when $\lambda \geqslant 1$). For compactness of the result, it is more convenient to express the stretch of the central bar:

$$\lambda_c = \frac{\sqrt{\lambda_s^2 - \sin^2\alpha^0}}{\cos\alpha^0}.$$

Substituting this expression in equation (4.1), we obtain a nonlinear equation from which we can calculate λ_s:

$$\sigma_c(\varepsilon_c)\Big|_{\varepsilon_c = \frac{\sqrt{\lambda_s^2-\sin^2\alpha^0}}{\cos\alpha^0}-1}\left(1-\nu_c\Big(\frac{\sqrt{\lambda_s^2-\sin^2\alpha^0}}{\cos\alpha^0}-1\Big)\right)^2 F_c^0 + \\ + 2\frac{\sigma_s(\varepsilon_s)\big|_{\varepsilon_s=\lambda_s-1}\sqrt{\lambda_s^2-\sin^2\alpha^0}\big(1-\nu_s(\lambda_s-1)\big)^2}{\lambda_s}F_s^0 = P. \quad (4.2)$$

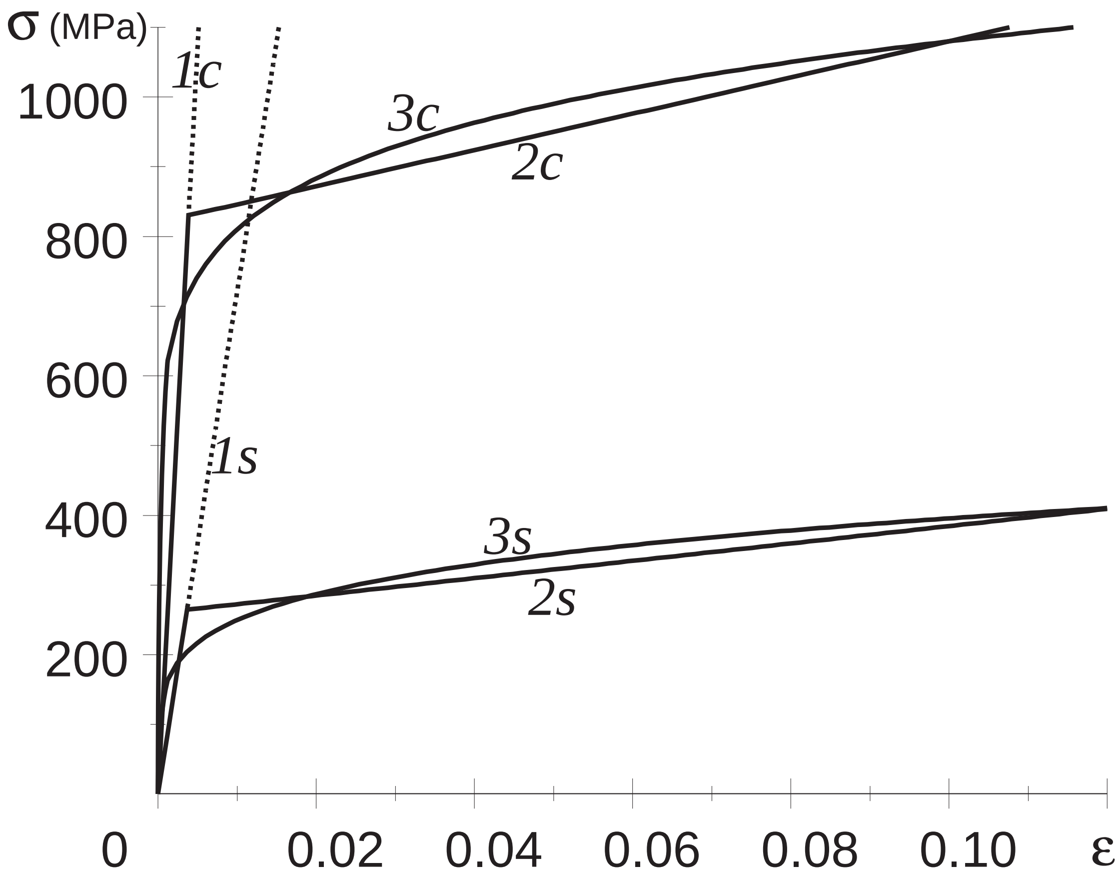

Fig. 3. The stress-strain relations of the materials 30HGSA and D16

Let us see how the left side of this equation behaves. For example, let us set the cross-sectional areas of all the bars to 1 cm^2, and the materials: the 30HGSA steel in the central bar, and the D16 aluminium alloy in the side bars. As a model of the materials, we consider a linearly elastic $\sigma = E\varepsilon$ (with the values $E_c = 215\,\text{GPa}$ and $E_s = 72\,\text{GPa}$, and also the plastic one: with the linear hardening (passing through the points $\{0, 0\}$, $\{\sigma_t/E, \sigma_t\}$, $\{\varepsilon_b, \sigma_b\}$ with the values $\sigma_{tc} = 830\,\text{MPa}$, $\sigma_{bc} = 1080\,\text{MPa}$, $\varepsilon_{bc} = 0.1$, $\sigma_{ts} = 265\,\text{MPa}$, $\sigma_{bs} = 410\,\text{MPa}$, $\varepsilon_{bs} = 0.12$), and with the power approximation ($\sigma_c = 1440\varepsilon_c^{1/8}$, $\sigma_s = 627\varepsilon_s^{1/5}$). The corresponding graphs are shown on figure 3. The Poisson's ratio for both the materials is given $\nu = 0.3$. Fig. 4 demonstrates the variants of behavior of the left-hand side of equation (4.2). Numbers 1, 2, 3 indicate the left-hand side graphs for the linearly elastic, linearly hardening, and power-law models of materials, respectively. Numbers 1a, 2a, 3a indicate similar graphs for the case when the reduction in the cross-sectional area of the rods is not taken into account when they are stretched (by assignment $\nu_c = \nu_s = 0$). It can be seen that if we neglect the decrease in the areas of the bar cross-sections under tension, then the behavior of the left side of equation (4.2) is monotonic, having a unique solution for any level

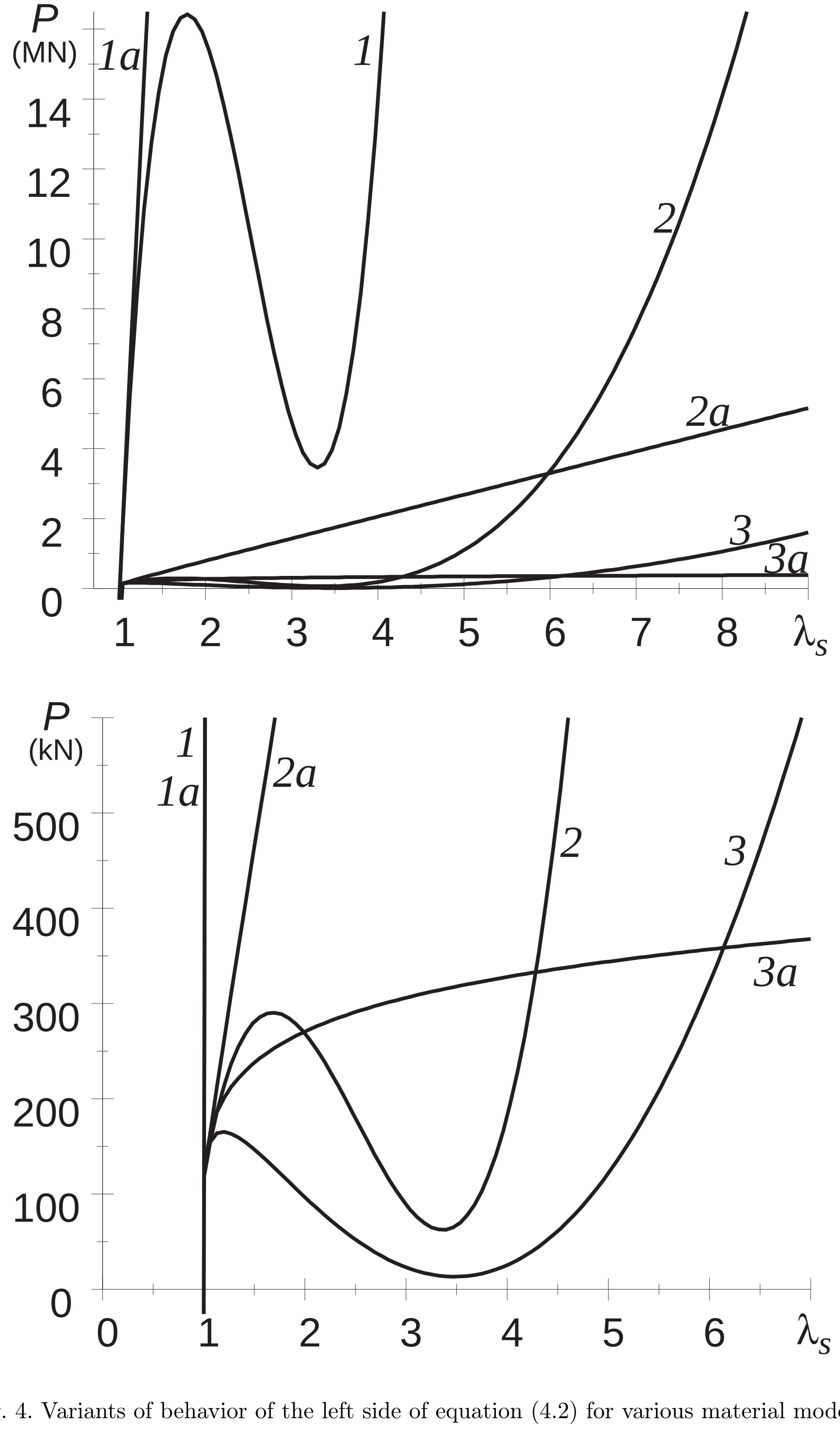

Fig. 4. Variants of behavior of the left side of equation (4.2) for various material models

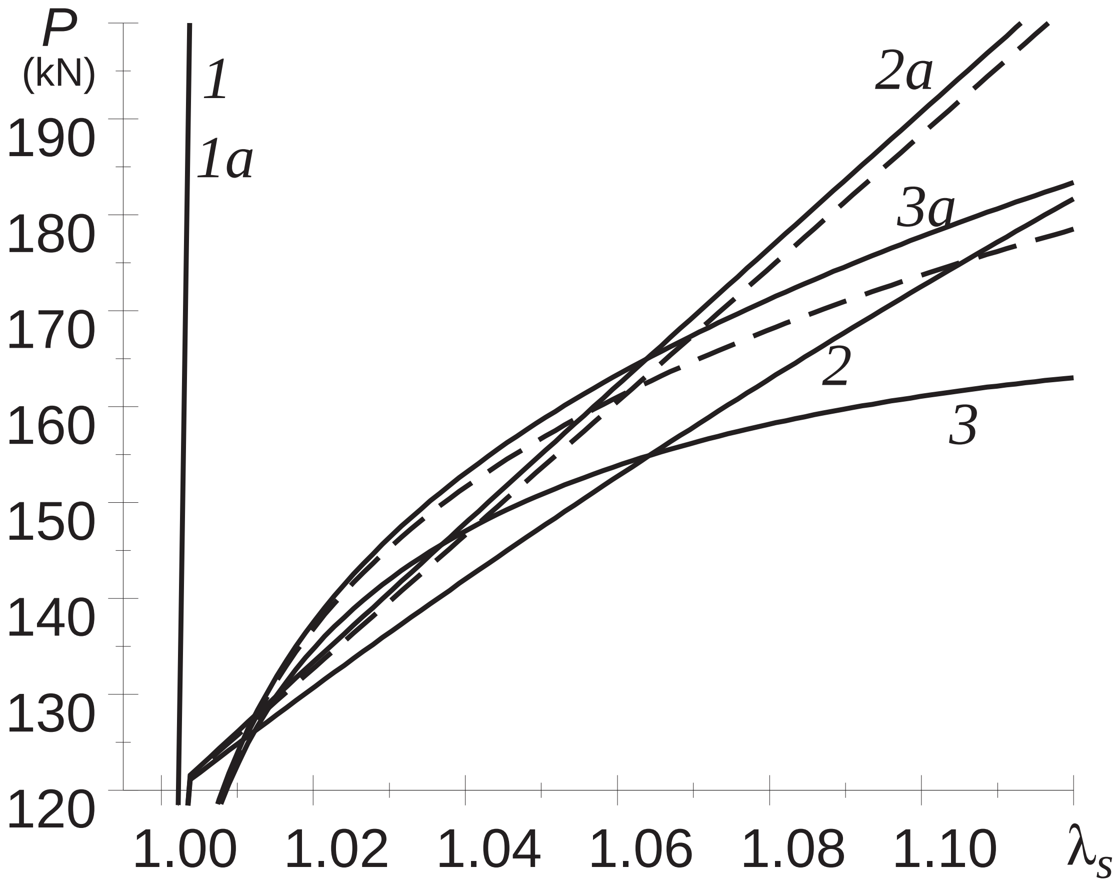

Fig. 5. Variants of behavior of the left side of equation (4.2) for various material models within the limits of strength

of external load. Nevertheless, when taking into account the true stresses, even with a linear-elastic model of materials, there is a range of values of the external load P, in which the deformed configuration of the truss is not the only one (three solutions become possible). However, the domain of non-uniqueness is beyond the real strength of the material used; therefore, it is of no practical interest. Within the limits of strength, behavior of the left-hand side of the equation written for all material models (it is shown in figure 5 by solid lines) is monotonic and gives the equation a unique solution. Although the curves in this area are rather close to each other, it is clear that neglecting the transverse compression of the rods results in an underestimation of the strain level near the ultimate strength by about 40%. It should also be noted that when using non-smooth approximations for the stress-strain curve, the equation solution graph will also have kinks, so when solving the equation by numerical methods one should use methods that do not use the derivative.

For small deformations, equation (4.1) becomes

$$\sigma_c F_c^0 + 2\sigma_s F_s^0 \cos\alpha^0 = P. \tag{4.3}$$

This equation, together with the condition of compatibility of small deformations (3.9), form a set of equations, similarly to the set (4.1), (3.7) for large strain. The conclusions obtained for the set (4.1), (3.7) are also valid for this set (except for the use of the concept of stretch λ). After eliminating the value ε_c from (4.3) (by the use of (3.9)), we obtain a nonlinear equation with respect to ε_s. This equation is in fact a weighted sum of stress-strain diagrams of the materials used. The behavior of the left side of this equation is shown vs the scale of stretches in Fig. 5 by dashed lines. It can be seen that a geometrically linear solution is even slightly more accurate than a solution for finite deformations without taking into account the true stresses.

Being considered for the linear case, both geometrically and physically, the system (3.9), (4.3) allow us to obtain explicit expressions for unknown parameters via given ones [3].

5. The formulation of the optimal design problem

Consider the problem of optimization the symmetric statically loaded three-bar truss, described above (see Fig. 1). It is required to find the values of the cross-sectional area of the bars, which ensure the minimum of mass for the truss under constraints from above on the stress levels in rods and the tecnology constraints. The design parameters are F_c^0 and F_s^0 — the cross-sectional areas of the rods of each type. The mass value of the truss is obviously expressed in terms of the design variables:

$$m = \rho_c F_c^0 l_c^0 + 2\rho_s F_s^0 l_s^0 = l_c^0 \Big(\rho_c F_c^0 + \frac{2\rho_s}{\cos\alpha^0} F_s^0 \Big)$$

where ρ_c, ρ_s are density of the materials used (we neglect the change in density during deformation).

Thus, the optimization problem is formulated as follows:

$$\begin{cases} \rho_c F_c^0 + \dfrac{2\rho_s}{\cos\alpha^0} F_s^0 \to \min\limits_{F_c^0, F_s^0} \\ \quad \sigma_c < \bar{\sigma}_c; \qquad\qquad \sigma_s < \bar{\sigma}_s \\ \quad \sigma_c = f_c(\lambda_c); \qquad\qquad \sigma_s = f_s(\lambda_s) \\ \sigma_c (1 - \nu\varepsilon_c)^2 F_c^0 + 2\sigma_s \dfrac{\lambda_c}{\lambda_s} (1 - \nu\varepsilon_s)^2 F_s^0 \cos\alpha^0 = P \quad \text{where} \quad \varepsilon = \lambda - 1 \\ \quad \lambda_s^2 = \lambda_c^2 \cos^2\alpha^0 + \sin^2\alpha^0 \\ \quad F_c^0 \geqslant \bar{F}_c; \qquad\qquad F_s^0 \geqslant \bar{F}_s \end{cases}$$

We can exclude dependent variables from the problem, and taking into account the fact that the dependences $\sigma = f(\lambda)$ are assumed to be monotonically nondecreasing, we can write:

$$\begin{cases} \rho_c F_c^0 + \dfrac{2\rho_s}{\cos\alpha^0} F_s^0 \to \min\limits_{F_c^0, F_s^0} \\ \lambda_c < f_c^{-1}(\bar{\sigma}_c); \qquad \lambda_c < \dfrac{\sqrt{\left(f_s^{-1}(\bar{\sigma}_s)\right)^2 - \sin^2\alpha^0}}{\cos\alpha^0} \\ g_c(\lambda_c) F_c^0 + 2g_s\left(\sqrt{\lambda_c^2\cos^2\alpha^0 + \sin^2\alpha^0}\right) F_s^0 \cos\alpha^0 = \dfrac{P}{\lambda_c} \\ \qquad\qquad \text{where} \quad g(\lambda) = \dfrac{f(\lambda)}{\lambda}(\lambda\nu - \nu - 1)^2 \\ F_c^0 \geqslant \bar{F}_c; \qquad F_s^0 \geqslant \bar{F}_s \end{cases} \tag{5.1}$$

For the case of small deformations, the problem of optimizing the truss is conveniently formulated using the formulation of the stress-strain relation through the secant modulus of elasticity ($\sigma = E^{\text{sec.}}\varepsilon$):

$$\begin{cases} \rho_c F_c^0 + \dfrac{2\rho_s}{\cos\alpha^0} F_s^0 \to \min\limits_{F_c^0, F_s^0} \\ \sigma_c < \bar{\sigma}_c; \qquad \sigma_s < \bar{\sigma}_s \\ \sigma_c = E_c^{\text{sec.}}\varepsilon_c; \qquad \sigma_s = E_s^{\text{sec.}}\varepsilon_s \\ \sigma_c F_c^0 + 2\sigma_s F_s^0 \cos\alpha^0 = P \\ \dfrac{\varepsilon_s}{\varepsilon_c} = \cos^2\alpha^0 \\ F_c^0 \geqslant \bar{F}_c; \qquad F_s^0 \geqslant \bar{F}_s \end{cases}$$

It follows from the two bottom equalities that

$$\varepsilon_c(E_c^{\text{sec.}} F_c^0 + 2E_s^{\text{sec.}} F_s^0 \cos^3\alpha^0) = P,$$

and the optimization problem can be written in the form:

$$\begin{cases} \rho_c F_c^0 + \dfrac{2\rho_s}{\cos\alpha^0} F_s^0 \to \min\limits_{F_c^0, F_s^0} \\ E_c^{\text{sec.}} F_c^0 + 2E_s^{\text{sec.}} F_s^0 \cos^3\alpha^0 > \dfrac{P}{\bar{\sigma}_c} E_c^{\text{sec.}} \\ E_c^{\text{sec.}} F_c^0 + 2E_s^{\text{sec.}} F_s^0 \cos^3\alpha^0 > \dfrac{P}{\bar{\sigma}_s} E_s^{\text{sec.}} \cos^2\alpha^0 \\ F_c^0 \geqslant \bar{F}_c; \qquad F_s^0 \geqslant \bar{F}_s \end{cases}$$

It can be seen from the above formulations that, for both small and finite deformations, the optimization problem of the considered truss is a linear programming problem. Accounting for each type of nonlinearity only affects change in scale of calibration of the coordinate axes. The acceptable region of the design variables and the

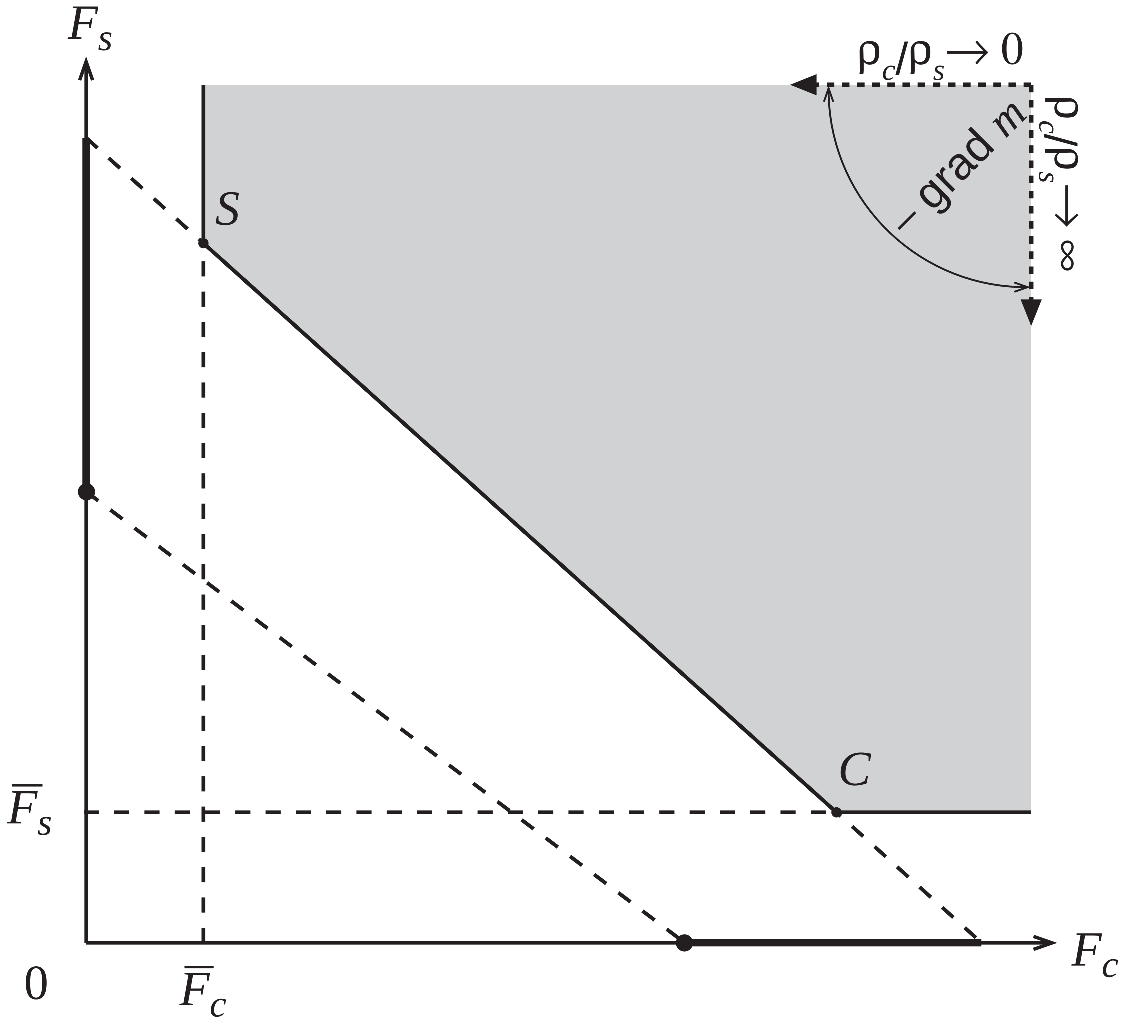

Fig. 6. The feasible region and the range of the objective function gradient

range of possible directions for the gradient of the objective function for this problem are shown in Fig. 6. Between the constraints on allowable stresses, one is active (i.e. is the strict equality), and the other is passive (being a strict inequality). Also, another low-probability event is also possible when the strains corresponding to the limiting stress levels of the rod materials satisfy the compatibility condition (3.7). Then both the stress limits will be active simultaneously, and in Fig. 6, their lines will coincide. Under zero constructive constraints, the formulation is beyond the linear programming problem, since one of the segments of the coordinate axes (shown in Fig. 6 by the thick line) should be added to the acceptable domain (this is due to the fact that in a degenerate rod it is no longer necessary to impose the stress constraint and the strain compatibility condition). We assume the technology constraints $\bar{F}$ to be non-zero but small enough that the active stress constraint does not overlap. Thus, in order to find the optimal design for the truss it is sufficient to consider only two solutions indicated in the figure by the letters C (which have $F_c^0 > \bar{F}_c$, $F_s^0 = \bar{F}_s$) and S ($F_c^0 = \bar{F}_c$, $F_s^0 > \bar{F}_s$): one of them having a smaller mass is the optimal design. Also it is possible (but improbable) the case when the gradient of the objective function is orthogonal to the active stress constraint, and the designs C and S have an equal mass. Then the optimal solution is any design belonging to the segment CS.

6. Influence of nonlinearities on the properties of the optimal design

Let us consider how the appearance and properties of the optimal design are changed while taking into account the physical and geometric nonlinearities. Restricted to small deformations, simple examples [8] (such as pure bending or torsion of a beam, loading of the considered here three-bar truss) clearly demonstrate that when considering the actual plastic behavior of a material, the internal forces acting in it are distributed more evenly compared to the linearly elastic case. As a result, accounting for the physical nonlinearity shows the increased load-carrying capacity of the structure. Thereby, one can meet assertions that a design based on the linear-elastic model of the material should result in an increase of the safety margin, and, in turn, the development taking into account the plasticity of the material should reduce the mass of the optimal design. For example, “design taking into account nonlinear characteristics of the material allows you to achieve an essential reduction in material consumption” [2]; “Elastic calculation leaves unused reserves of load-carrying capacity in statically redundant systems. The involvement of these reserves in the work, associated with significant savings in building materials, is only possible with the moving to calculation taking into account the plastic work of the material” [6]. It also notes that “The actual effect of the transition from conventional to ‘plastic’ frame design is expressed in 25-30% of the metal savings.” However, the simple models used as a basis for these observations contain only one structural material. If more than one material is used then an additional degree of freedom arises related to the ratio of the densities of the materials used. This degree of freedom can result in the opposite effect. As for the nature of the influence of accounting for geometric nonlinearity on the properties of optimal designs, there are no simple examples that allow to draw unambiguous conclusions. With all this, it is obvious that

Table 1. The properties of the used materials

Material	ρ (kg/m^3)	E (GPa)	σ_t (MPa)	σ_b (MPa)	ε_b (%)
Bronze BrO10	8800	104	175	215	7
Brass L75	8630	103	110	370	60
Steel 30HGSA	7850	215	830	1080	10
Cast iron SCh35	7400	140	—	350	σ_b/E
Titanium alloy VT6	4450	115	1030	1080	6
Aluminium alloy D16	2770	72	265	410	12
Aluminium alloy ML5	1810	43	90	160	2

the simultaneous consideration of the physical and geometric nonlinearities should lead to the most realistic and adequate design.

Several standard structural alloys [5] was selected for the investigation. Their parameters are given in Table 1. The properties of optimal designs were analyzed with all possible combinations of materials use, i.e. altogether 49 possible options. The optimization problem (5.1) was considered with the following parameters: $l_c^0 = 1$ m, $\alpha^0 = \pi/4$, $P = 10$ kN, $\bar{F}_c = \bar{F}_s = 1$ mm^2, the strength limits $\bar{\sigma}$ was specified by the ltimate stresses σ_b. To describe the geometrically nonlinear behavior of materials, a model with the linear hardening was used (except for the cast iron, which, through its brittleness, was considered to be linearly elastic in all calculations).

Combinations of materials, giving rise to the minimum or maximum mass of the optimal design, are shown in Table. 2 (in this and the following tables, the used materials are enumerated with specifying the material of the central bar, and then through the dash, the material of the side rods). From the changes in the set of materials, it can be seen that the physical nonlinearity has a greater effect on the change in the properties of the optimal design than the geometric one. Wherein, even though as expected on the basis of simple examples, use of physical nonlinearity results in a decrease in the mass of the optimal design, optimal designs with the largest mass were also obtained taking into account the physical nonlinearity.

Table 3 analyzes the nature of the changes in the mass of the truss at accounting for each of the kinds of nonlinearity. In particular, taking into account the physical nonlinearity leads to an unambiguous reduction in the mass of the optimal design only when one structural material is used in all the rods. Meanwhile, already for two materials, the cases of increasing as well as decreasing mass value appear. In addition, in seven cases the mass and other parameters of the optimal design did not change (all these cases are connected with the use of non-plastic cast iron in the fully stressed element (i.e. the bar with an active stress constraint) and the compatible stress level below the plasticity zone in the understressed element; among them there is a rather strange design VT6–SCh35 in which the fully stressed element is simultaneously

Table 2. Optimal designs having the smallest and largest mass value

Taking into account nonlinearities	Smallest mass			Largest mass		
	materials	design	m (g)	materials	design	m (g)
Lin. elast., small def.	VT6–D16	C	47	BrO10–BrO10	C	428
Plast., small def.	VT6–ML5	C	46	L75–BrO10	C	516
Lin. elast., fin. def.	VT6–D16	C	47	BrO10–BrO10	C	428
Plast., fin. def.	VT6–ML5	C	46	L75–BrO10	C	528
including for the one-material truss						
Lin. elast., small def.	VT6–VT6	C	51	BrO10–BrO10	C	428
Plast., small def.	VT6–VT6	C	48	BrO10–BrO10	C	423
Lin. elast., fin. def.	VT6–VT6	C	51	BrO10–BrO10	C	428
Plast., fin. def.	VT6–VT6	C	48	BrO10–BrO10	C	425

Table 3. Change in the mass of the optimal project at accounting for the nonlinearities

Nonlinearity under study	other	$m_{\text{lin.}} > m_{\text{nonl.}}$ cases	Δm_{max}	$m_{\text{lin.}} = m_{\text{nonl.}}$ number of cases	$m_{\text{lin.}} < m_{\text{nonl.}}$ number of cases	Δm_{max}
Physical	small def.	27	355%	7	15	51%
	fin. def.	25	359%	7	17	52%
Geometric	lin. elast.	1	0.03%	0	48	0.22%
	plast.	14	1.57%	0	35	8.3%
including for the one-material truss						
Physical	small def.	6	6.2%	1	0	—
	fin. def.	5	6.3%	1	1	7.6%
Geometric	lin. elast.	0	—	0	7	0.04%
	plast.	1	0.06%	0	6	8.3%

prone to degeneration). The effect of increasing the mass of the optimal design, while taking into account the physical nonlinearity, was manifested for 15 combinations of the materials used, as can be seen from Table 3. In 7 cases (from these 15) this effect is the most appreciable (weight increase is greater than 2.5%). The corresponding designs are shown in Table 4. Also, tables 3 and 4 make it possible to note that an additional account for geometric nonlinearity has a weak influence on the quantitative trend of the mass change. Nevertheless, here it is possible to increase the mass of a physically nonlinear design from one material (L75–L75), which is not obtained from the calculation on the basis of small deformations and does not confirm the analysis of the simplest examples.

It was analyzed how much the designs C and S differ in their mass for the same problem, and how realistic in practice is the optimality of the entire segment CS. The ranges in which there is a difference in mass between these designs are shown in Table 5. As a matter of fact, both for small and finite deformations, linearly elastic calculation gives the optimal design C in all 49 cases. However, when physically nonlinear calculation was used, design S was already optimal in 11 cases. This demonstrates the fundamental possibility of changing not only numerical parameters, but also the

Table 4. Optimal designs with increasing mass when allowance for physical linearity

Used materials	Small deformations Lin.-elast.	Small deformations Plastic	Δm	Finite deformations Lin.-elast.	Finite deformations Plastic	Δm
L75–BrO10	C (252 g)	C (516 g)	51%	C (252 g)	C (528 g)	52%
L75–SCh35	C (246 g)	S (427 g)	42%	C (246 g)	S (426 g)	42%
ML5–SCh35	C (130 g)	C (191 g)	32%	C (130 g)	C (191 g)	32%
D16–ML5	C (72 g)	C (92 g)	23%	C (94 g)	C (120 g)	21%
30HGSA–SCh35	C (90 g)	C (111 g)	18%	C (90 g)	C (111 g)	18%
30HGSA–ML5	C (77 g)	C (88 g)	13%	C (77 g)	C (89 g)	13%
L75–L75	C (252 g)	C (250 g)	–0.7%	C (252 g)	C (272 g)	8%

Table 5. Difference in mass between the designs C and S (in % of m_{opt})

Accounting of nonlinearities	min. difference materials	opt.	Δm	max. difference materials	opt.	Δm
Lin. elast., small. def.	L75–30HGSA	C	64.9	D16–BrO10	C	592
Plast., small. def.	30HGSA–BT6	C	10.4	VT6–L75	C	1843
Lin. elast., finite def.	L75–30HGSA	C	64.6	D16–BrO10	C	591
Plast., finite. def.	30HGSA–VT6	C	9.7	VT6–L75	C	1822
including for the one-material truss						
Lin. elast., small. def.	30HGSA–30HGSA	C	199	BrO10–BrO10	C	276
Plast., small. def.	VT6–VT6	C	67.9	SCh35–SCh35	C	263
Lin. elast., finite def.	VT6–VT6	C	198	ML5–ML5	C	282
Plast., finite. def.	VT6–VT6	C	67	SCh35–SCh35	C	262

geometry of the optimal design, taking into account the physical nonlinearity. The specified cases of changing the geometry are shown in the Table 6. It can be seen from the table that in all these cases the linear design has its technology constraints active in the side bars, and the nonlinear one has them active in the central bar. At the same time, in the linear case, the stress constraint is active only in the central bar, but in the nonlinear case it is active in the central bar in 4 cases (are abnormal projects in which the fully stressed element is simultaneously prone to degeneration) and in 7 cases it is active in the side bars. Qualitatively, the same results were obtained for finite deformations (in the table a comma-separated value of the mass relates to the corresponding geometrically nonlinear design, if it differs from the geometrically linear case). Thus, the physical nonlinearity has a greater impact on the change in the

Table 6. Optimal designs with a modified geometry after taking into account the physical nonlinearity

Materials	Optimal design: Linear-elastic (small and finite def.)	Optimal design: Plastic (small, finite def.)
BrO10–SCh35	C (422 g)	S (424 g)
BrO10–30HGSA	C (419 g)	S (177 g, 175 g)
BrO10–D16	C (413 g)	S (187 g, 185 g)
BrO10–ML5	C (412 g)	S (231 g, 229 g)
BrO10–VT6	C (415 g)	S (91 g, 90 g)
ML5–VT6	C (122 g)	S (87 g, 86 g)
L75–SCh35	C (246 g)	S (427 g, 426 g)
L75–30HGSA	C (243 g)	S (151 g, 149 g)
L75–D16	C (237 g)	S (141 g, 139 g)
L75–ML5	C (236 g)	S (232 g, 231 g)
L75–VT6	C (239 g)	S (90 g, 89 g)

configuration of the optimal design than the geometric one.

The existence of unusual designs having the fully stressed element, which is prone to degeneration, can be explained by the fact that the other (underloaded) element made of material with large allowable stress. And, despite its underload, the last element has the stress level higher than that in the fully stressed one; i.e., this effect is due to the use of different materials in the one structure.

7. Conclusion

In spite of its simplicity, the problem of optimal design of a three-bar truss can exhibit quite interesting effects when geometric and physical nonlinearities are taken into account. In general, the geometric nonlinearity affects the changes in the optimal design appreciably weaker than the physical one. This is the most evident especially in calculations performed on the basis of a linearly elastic material model. In this structure, geometrically nonlinear effects can appear more significant while using rubberlike materials which allow to achieve large deformations within the limits of their strength.

References

1. *Аоки М.* Введение в методы оптимизации: Пер. с англ. — М: Наука, 1977. — 344 с.

Aoki M. Introduction to Optimization Techniques: Fundamentals and Applications of Nonlinear Programming. New York: Macmillan Company, 1971.

2. *Баничук Н. В.* Введение в оптимизацию конструкций. — М: Наука, 1986. — 304 с.

Banichuk, N. V. Introduction to optimization of structures. New York: Springer, 1990.

3. *Биргер И. А., Мавлютов Р. Р.* Сопротивление матиреалов: Учебн. пособие. — М: Наука, 1986. — 560 с.

Birger I. A., & Mavlyutov R. R. Strength of materials. Moscow: Fizmatlit, 1986. (in Russian).

4. *Липин Е. К., Фролов В. М., Чедрик В. В., Шаныгин А. Н.* Алгоритм оптимизации силовых конструкций по условиям прочности с компенсацией нарушенных ограничений // Ученые записки ЦАГИ. — 1988. — Т. XIX, №1. — С. 58–66.

Lipin, E. K., Frolov, V. M., Chedrik, V. V., & Shanygin, A. N. An algorithm of optimization of load-bearing structures relative to the strength conditions with compensation for violated constraints, Uchen. Zap. TSAGI, 19, No. 1, 58–66 (1988). (in Russian).

5. *М*арочник стали и сплавов. — `http://www.splav-kharkov.com/choose_type.php`.

The marker of steel and alloys `http://www.splav-kharkov.com/choose_type.php` Accessed 27.12.2017 (in Russian).

6. *С*правочник проектировщика промышленных, жилых и общественных зданий и сооружений расчётно-теоретический. Ред. Уманский А.А. — М: ГСИ, 1967. — 1046 с.

Ed. Umansky A. A. The calculational and theoretical handbook for a designer of industrial, residential and public buildings and structures. Moscow: GSI, 1967. (in Russian).

7. *Хог. Э., Арора Я.* Прикладное оптимальное проектирование: Механические системы и конструкции: Пер. с англ. — М: Мир, 1983. — 478 с.

   Haug. E.J., Arora J.S. Applied optimal design: mechanical and structural systems. New York: Wiley Interscience, John Wiley and Sons, 1979.

8. *Ходж Ф. Г.* Расчет конструкций с учетом пластических деформаций: Пер. с англ. — М: Машгиз, 1963. — 380 с.

   Hodge Ph. G., Jr. Plastic analysis of structures. New York: McGraw-Hill Book Co, 1959.

9. *Greco M., Gesualdo F. A. R., Venturini W. S., Coda H. B.* Nonlinear positional formulation for space truss analysis. Finite Elements in Analysis and Design, 42, No. 12, 1079–1086 (2006).

10. *Hodge P. G., Bathe K. J., Dvorkin E. N.* Causes and Consequences of Nonuniqueness in an Elastic-Perfectly-Plastic Truss. Journal of Applied Mechanics, 53, 235—241 (1986).

11. *Ryu Y. S., Haririan M., Wu C. C., Arora J. S.* Structural design sensitivity analysis of nonlinear response. Computers & Structures, 21, No. 1/2, 2045–2055 (1985).

12. *Selyugin S. V., Chekhov V. V.* Multimaterial design of physically nonlinear structures. Structural and Multidisciplinary Optimization, 21, No. 3, 209–217 (2001).

13. *Yang W. H.* Large deformation of structures by sequential limit analysis. International Journal of Solids and Structures, 30, No. 7, 1001–1013 (1993).

Получена 20.06.2017