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On periodic translations on n-torus[1]

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Abstract. We consider periodic translations on n-torus and investigate the set of all conjugating homeomorphisms for topologically conjugated translations. It was shown by J. Nielsen [Dansk Videnskaternes Selskab. Math.-fys. Meddelerer, 1937, Vol.15, 1-77] that two periodic homeomorphisms of two-torus such that all points have the same period are topologically conjugate if and only if they have the same period. We consider the problem when two periodic translations on n-torus are topologically conjugate by means of toral automorphism. The main result is that two periodic translations on n-torus of the same period are topologically conjugate by means of countable family of toral automorphisms. Moreover, we show that for two periodic translations that are topologically conjugate each homotopy class in the set of all conjugating homeomorphisms contains continuum of homeomorphisms.

Keywords: topological conjugacy, toral automorphism, homotopy.

1. Introduction and statement of results

Periodic translations of the n-dimensional ($n \geqslant 2$) torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ are studied in this paper. The transformation $f\colon \mathbb{T}^n \to \mathbb{T}^n$ is called periodic of period k, if $f^k = id$, and for any $k' < k$ the inequality $f^{k'} \neq id$ holds.

Periodic translations of two-dimensional surfaces were considered in detail by J. Nielsen [4]. Generally speaking, periodic translations of period k may have points of a period less than k. However, orientation-preserving periodic homeomorphisms have only a limited number of points of a period less than k, while all other points have the same period k.

In this article we study periodic n-dimensional translations such that all points have the same period, that are described by a shift to an n-dimensional vector with rational coordinates. The transformations $f\colon \mathbb{T}^n \to \mathbb{T}^n$ have the following form:

$$f(x) = x + \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix} \mod 1. \tag{1.1}$$

It is known (see, for example, [3]) that depending on the value of γ_i, the transformation f can have the following type:

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1. If all γ_i are rational, then all the points of f have a period equal to the lowest common denominator of all γ_i;
2. If γ_i are linearly independent over the field $\mathbb{Q}$, then f is topologically transitive and the trajectory of each point $x \in \mathbb{T}^n$ is dense in $\mathbb{T}^n$;
3. If γ_i are linearly dependent over $\mathbb{Q}$, but not all γ_i are rational, then the closure of the trajectory of any points of $x \in \mathbb{T}^n$ is a finite union of k-dimensional tori $\mathbb{T}^k$, where $1 \leqslant k \leqslant n-1$.

We denote by G_n the set of all periodic homeomorphisms of the n-dimensional torus of the form (1.1).

Nielsen [4] had shown that two periodic homeomorphisms of the two-dimensional torus $\mathbb{T}^2$ such that all points have the same period are topologically conjugate if and only if their periods coincide. However, the problem whether two homeomorphisms of a torus are conjugate by means of a group automorphism of the torus was not considered. In this article we study the problem when two transformations from the class G_n are topologically conjugate by means of the group automorphism of the torus $\mathbb{T}^n$.

The main result of the article is presented in the following theorems.

Theorem 1. *If two periodic homeomorphisms of the n-torus $f\colon \mathbb{T}^n \to \mathbb{T}^n$ and $f'\colon \mathbb{T}^n \to \mathbb{T}^n$, $f, f' \in G_n$ have the same period, then there exists a countable family $\{h_i\}, i \in \mathbb{N}$ of group automorphisms of the n-torus $h_i\colon \mathbb{T}^n \to \mathbb{T}^n$ conjugating the maps f and f'.*

From the results of D. Z. Arov [1] it follows that if two transitive translations of the n-torus are topologically conjugate, then the conjugating homeomorphism must be linear (a composition of a group automorphism and a shift). The opposite result holds for periodic shifts.

Theorem 2. *If two periodic homeomorphisms of the torus $f\colon \mathbb{T}^n \to \mathbb{T}^n$ u $f'\colon \mathbb{T}^n \to \mathbb{T}^n$, $f, f' \in G_n$ are topologically conjugate by means of homeomorphism $h\colon \mathbb{T}^n \to \mathbb{T}^n$, then there is a continuum set of homeomorphisms of an n-dimensional torus $\{h_\beta\}, \beta \in A$, $h_\beta\colon \mathbb{T}^n \to \mathbb{T}^n$ homotopic to h such that $h_\beta \circ f = f' \circ h_\beta$.*

2. Proof of the results

We denote the greatest common divisor of the set of integers $a_1, a_2, \ldots, a_n$ by $(a_1, a_2, \ldots, a_n)$.

Lemma 1. *For any pair of n-dimensional vectors*

$$v_1 = \begin{pmatrix} 1/q \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad u \quad v_2 = \begin{pmatrix} p_1/q \\ p_2/q \\ \vdots \\ p_n/q \end{pmatrix}, \quad q \in \mathbb{N}, \quad p_i \in \mathbb{Z},$$

such that $(p_1, p_2, \ldots, p_n, q) = 1$ there exists an unimodular matrix A and integers $m_1, m_2, \ldots, m_n$, such that an equality $Av_1 = v_2 + (m_1, m_2, \ldots, m_n)^T$ holds.

Proof. Suppose that the matrix A has the form

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}.$$

Let's set the elements of the matrix A and numbers $m_1, m_2, \dots, m_n$, in such a way that they satisfy the conclusion of the lemma. Let elements a_{i1}, $i = \overline{1,n}$ be of the form $a_{i1} = p_i + qm_i$. We denote the complement minor of the element a_{i1} by M_i. Let us show that the numbers $m_1, \dots, m_n$ can be chosen in such a way that $(p_1+qm_1, \dots, p_n+qm_n) = 1$. We introduce the notation $r_i = (p_i, q)$, $R = \max\limits_i r_i$. Since p_i/r_i and q/r_i are relatively prime, it follows from the Dirichlet theorem (see, for example, [2]) that any arithmetic progression with coprime the first term and the difference contains infinitely many prime terms. That is why, the sequence $\frac{p_i}{r_i} + \frac{q}{r_i}m_i$, where $\frac{p_i}{r_i}$ and $\frac{q}{r_i}$ are fixed, and m_i are different integer values, contains infinitely many simple terms. We choose m_i in such a way that $\frac{p_i}{r_i} + \frac{q}{r_i}m_i$ will be different prime numbers exceeding in absolute value R. If we chose m_i in such a way, then an equality $(a_{11}, \dots, a_{n1}) = 1$ holds, since $(r_1, \dots, r_n) = 1$.

Expanding $det(A)$ by the first column, we note that $det(A) = \sum_{i=1}^{n}(-1)^{1+i}a_{i1}M_i$. We will prove the lemma if we can choose integer elements a_{ij}, $i = \overline{1,n}$, $j = \overline{2,n}$ in such a way, that the corresponding values of M_i, $i = \overline{1,n}$, will be such that $\sum\limits_{i=1}^{n}(-1)^{1+i}a_{i1}M_i = 1$.

For this, we show that we can choose the corresponding integer values a_{ij}, $i = \overline{1,n}$, $j = \overline{2,n}$ for any fixed set of integer values M_i^* of complement minors M_i, $i = \overline{1,n}$. Let matrix A be of the form (2.1), i.e. a matrix whose $a_{ij} = 0$, $i - j \geqslant 1$, $j \neq 1$.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{24} & \dots & a_{1n-1} & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & \dots & a_{2n-1} & a_{2n} \\ a_{31} & 0 & a_{33} & a_{34} & \dots & a_{3n-1} & a_{3n} \\ a_{41} & 0 & 0 & a_{44} & \dots & a_{4n-1} & a_{4n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-11} & 0 & 0 & 0 & \dots & a_{n-1n-1} & a_{n-1n} \\ a_{n1} & 0 & 0 & 0 & \dots & 0 & a_{nn} \end{pmatrix}. \tag{2.1}$$

We prove this statement, using mathematical induction on n. The statement is obvious for $n = 2$, since the minors M_1, M_2 coincide with the elements a_{12}, a_{22}. Suppose that it is true for $n = k$ and show that it is also true for $n = k + 1$. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{24} & \dots & a_{1k} & a_{1k+1} \\ a_{21} & a_{22} & a_{23} & a_{24} & \dots & a_{2k} & a_{2k+1} \\ a_{31} & 0 & a_{33} & a_{34} & \dots & a_{3k} & a_{3k+1} \\ a_{41} & 0 & 0 & a_{44} & \dots & a_{4k} & a_{4k+1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{k1} & 0 & 0 & 0 & \dots & a_{kk} & a_{kk+1} \\ a_{k+11} & 0 & 0 & 0 & \dots & 0 & a_{k+1k+1} \end{pmatrix}. \tag{2.2}$$

By $\widetilde{M_i}$ we denote the minors obtained from A by deleting the i-st and $(k+1)$-th rows and the 1-th and $(k+1)$-th column. Note that $M_i = a_{k+1k+1}\widetilde{M_i}$, $i = \overline{1,k}$, expanding M_i by the last line. We set $a_{k+1k+1} = (M_1^*, M_2^*, \ldots, M_k^*)$. By the induction hypothesis the numbers a_{ij}, $i-j \geqslant 0$, $i = \overline{2,k}$, $j = \overline{2,k}$ can be chosen in such a way that $\widetilde{M_i} = \frac{M_i^*}{(M_1^*, M_2^*, \ldots, M_k^*)}$, $i = \overline{1,k}$.

Consider the minor M_{k+1}. We notice that $M_{k+1} = \sum\limits_{i=1}^{k}(-1)^{1+k}\widetilde{M_i}a_{ik+1}$, expanding M_{k+1} by the last column. We chose the elements in such a way that $(\widetilde{M_1}, \widetilde{M_2}, \ldots, \widetilde{M_k}) = 1$. For this reason we can choose the elements $a_{1k+1}, \ldots, a_{kk+1}$ in such a way that the expression $\sum\limits_{i=1}^{k}(-1)^{1+k}\widetilde{M_i}a_{ik+1}$ will take the value M_{k+1}^*. □

Let's an elementary lemma before proceeding to the proof of theorem 1.

Lemma 2. *Let A and B be unimodular $n \times n$ matrices, and $f\colon \mathbb{T}^n \to \mathbb{T}^n$ and $g\colon \mathbb{T}^n \to \mathbb{T}^n$ are the group automorphisms of the n-dimensional torus induced by them. Then the group automorphism of the n-dimensional torus induced by the product AB coincides with the composition of the maps $f \circ g$ and the group automorphism induced by the matrix A^{-1} coincides f^{-1}.*

Proof. Let $x \in \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, that is, $x = (x_1, \ldots, x_n)^T$, $0 \leqslant x_i < 1$. We set $Bx = y + v$, $0 \leqslant y_i < 1$, $v_i \in \mathbb{Z}$. Therefore we have $f \circ g(x) = A(Bx \mod 1) \mod 1 = A(y + v \mod 1) \mod 1 = A(Bx - v) \mod 1 = ABx - Av \mod 1 = ABx \mod 1$ by the definition of f and g, and taking into account the integer property of the elements of the matrix B. Let $h\colon \mathbb{T}^n \to \mathbb{T}^n$ be a group toral automorphism induced by the matrix A^{-1}, then, we have $f \circ h = h \circ f = id$ by virtue of the just proved, considering the product of matrices $AA^{-1} = A^{-1}A = E$. □

Lemma 3. *If two periodic homeomorphisms of n-torus $f\colon \mathbb{T}^n \to \mathbb{T}^n$ и $f'\colon \mathbb{T}^n \to \mathbb{T}^n$, $f, f' \in G_n$ have the same periods, then they are topologically conjugate by means of the group automorphism of the n-torus $h\colon \mathbb{T}^n \to \mathbb{T}^n$*[2].

Proof. Let q be the period of the maps f and f'. Then f and f' can be represented in the form $f(x) = x + u \mod 1$, $f(x) = x + u' \mod 1$, where

$$u = \begin{pmatrix} p_1/q \\ p_2/q \\ \vdots \\ p_n/q \end{pmatrix}, \quad u' = \begin{pmatrix} p_1'/q \\ p_2'/q \\ \vdots \\ p_n'/q \end{pmatrix}, \quad p_i,\ p_i' \in \mathbb{Z},$$

and $(p_1, p_2, \ldots, p_n, q) = 1$, $(p_1', p_2', \ldots, p_n', q) = 1$.

[2]Two dynamical systems $f\colon X \to X$ и $f'\colon X \to X$, defined on a topological space X , are topologically conjugate if there exists a homeomorphism $h\colon X \to X$, что $f \circ h = h \circ f'$.

Let $g\colon \mathbb{T}^n \to \mathbb{T}^n$ be a periodic toral homeomorphism of the following form $g(x) = x + v \mod 1$, where $v = (1/q, 0, \dots, 0)^T$

It follows from lemma 1 that there exist unimodular $n \times n$ matrices A and A' such that $Av \equiv u \pmod 1$, $A'v \equiv u' \pmod 1$. Let us show that the group toral automorphisms $h\colon \mathbb{T}^n \to \mathbb{T}^n$ and $h'\colon \mathbb{T}^n \to \mathbb{T}^n$ induced by the matrices A и A' are such that $f \circ h = h \circ g$ and $f' \circ h' = h' \circ g$. Fix the point $x \in \mathbb{T}^n$, $0 \leqslant x_i < 1$ and set $w = (x+v) - (x+v \mod 1)$. We have $h \circ g(x) = A(x+v \mod 1) \mod 1 = A(x+v-w) \mod 1 = Ax+Av+Aw \mod 1 = Ax+u \mod 1 = (Ax \mod 1+u) \mod 1 = f \circ h(x)$. Since the point x was chosen arbitrary, we have an equality $f \circ h = h \circ g$. The second equality is proved similarly. We note that the group automorphism induced by the matrix AA'^{-1} coincides with $h \circ h'^{-1}$ by the virtue of lemma 2. Now it is easy to see that the desired conjugation has the form $f \circ (h \circ h'^{-1}) = (h \circ h'^{-1}) \circ f'$. □

Lemma 4. *Let a periodic homeomorphism of an n-dimensional torus $g\colon \mathbb{T}^n \to \mathbb{T}^n$ of a period q has the form $g(x) = x + v$, where* $v = \begin{pmatrix} 1/q \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. *Then there exists a countable set of unimodular $n \times n$ matrices a_i such that the equality $g \circ h_i = h_i \circ g$ holds for the group automorphisms of the n-dimensional torus $h_i\colon \mathbb{T}^n \to \mathbb{T}^n$ induced by them.*

Proof. Let's denote a set of unimodular matrices such that the first column of the matrix A_i has the form $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ by $\{A_i\}, i \in \mathbb{N}$.

Fix the point $x \in \mathbb{T}^n$ and set $w = (x + v) - (x + v \mod 1)$. We have $h_i \circ g(x) = A_i(x+v \mod 1) \mod 1 = A_i(x+v-w) \mod 1 = A_ix+A_iv-A_iw \mod 1 = A_ix+v \mod 1 = (A_ix \mod 1 + v) \mod 1 = g \circ h_i(x)$ by the choice of A_i. The equality $h_i \circ g = g \circ h_i$ is true since the point x was chosen arbitrary. □

2.1. Proof of the theorem 1

Proof. Let q be the period of the maps f and f', and let $g\colon \mathbb{T}^n \to \mathbb{T}^n$ be the periodic homeomorphism defined in lemma 4. It follows from lemma 3 that there are group automorphisms $h\colon \mathbb{T}^n \to \mathbb{T}^n$ and $h'\colon \mathbb{T}^n \to \mathbb{T}^n$ induced by the matrices A and A', respectively, for which the equalities $h \circ f = g \circ h$, $h' \circ f' = g \circ h'$ hold. It follows from lemma 4 that there exists a countable family of unimodular matrices $\{A_i\}, i \in \mathbb{N}$ such that the group automorphisms $h_i\colon \mathbb{T}^n \to \mathbb{T}^n$ induced by them are pairwise non-homotopic, and the equalities $g \circ h_i = h_i \circ g$ hold. It follows from lemma 2 that the group automorphism induced by the matrix $A'^{-1}A_iA$ coincides with $(h'^{-1} \circ h_i \circ h)$ and the group automorphisms $h'^{-1} \circ h_i \circ h$ are pairwise non-homotopic, since h_i are pairwise non-homotopic. It is easy to see that the desired conjugation has the form $(h'^{-1} \circ h_i \circ h) \circ f = f' \circ (h'^{-1} \circ h_i \circ h)$. □

2.2. Proof of the theorem 2

Proof. Let q be the period of the maps f and f', and let $g\colon \mathbb{T}^n \to \mathbb{T}^n$ be the homeomorphism defined in lemma 4. We have $h \circ f = f' \circ h$ by the condition of the theorem. Since f' and g have the same period, then there is a homeomorphism $h'\colon \mathbb{T}^n \to \mathbb{T}^n$ such that the equality $h' \circ f' = g \circ h'$ holds. We consider the set $\{h_\beta\}$ of all possible homeomorphisms $h_\beta\colon \mathbb{T}^n \to \mathbb{T}^n$ of the form

$$h_\beta(x) = x + \begin{pmatrix} u_\beta(x_2, x_3, \ldots, x_n) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \mod 1,$$

where $u_\beta\colon \mathbb{T}^{n-1} \to \mathbb{R}$ is an arbitrary continuous function of $n-1$ variables, periodic by each argument. Ntote that for any h_β we have the equality $h_\beta \circ g = g \circ h_\beta$ and each homeomorphism $\{h_\beta\}$ is homotopic to the identical mapping. We set $h_\alpha = h'^{-1} \circ h_\beta \circ h' \circ h$. It is easy to see that for the homeomorphism $h'^{-1} \circ h_\beta \circ h' \circ h$ the equality $(h'^{-1} \circ h_\beta \circ h' \circ h) \circ f = f' \circ (h'^{-1} \circ h_\beta \circ h' \circ h)$ holds. Moreover, the homeomorphism $h'^{-1} \circ h_\beta \circ h' \circ h$ is homotopic to h, since the homeomorphism $h'^{-1} \circ h_\beta \circ h'$ is homotopic to the identical mapping. Thus, the set $\{h_\alpha\}$ is the desired family of conjugating homeomorphisms. □

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