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On obstructions to the existence of a simple arc, connecting the multidimensional Morse-Smale diffeomorphisms[1]

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Abstract. In this paper we consider Morse-Smale diffeomorphisms defined on a multidimensional nonsimply connected closed manifold $M^n, n \geq 3$. For such systems, the concept of trivial (nontrivial) connectedness of their periodic orbits is introduced. It is established that isotopic trivial and nontrivial diffeomorphisms can not be joined by an arc with codimension one bifurcations. Examples of such pair of Morse-Smale cascades on the manifold $\mathbb{S}^{n-1} \times \mathbb{S}^1$ are constructed.

Keywords: Morse-Smale diffeomorphisms, bifurcation, smooth arc

1. Introduction and a formulation of results

The present paper has deal with a solution of the Palis-Pugh problem [10] on the existence of an arc with a finite or countable set of bifurcations connecting two Morse-Smale systems on a smooth closed manifold M^n. S. Newhouse and M. Peixoto [8] proved that any Morse-Smale vector fields can be connected by a simple arc. Simplicity means that the arc consists of the Morse-Smale systems with the exception in a finite set of points in which the vector field deviates by at least way (in a certain sense) from the Morse-Smale system. Below we give a definition of the simple arc for discrete Morse-Smale systems.

Let $Diff(M^n)$ be the space of diffeomorphisms on a closed manifold M^n with C^1-topology and $MS(M^n)$ be the subset of Morse-Smale diffeomorphisms. *Smooth arc* in $Diff(M^n)$ is a smooth map

$$\xi \colon M^n \times [0,1] \to M^n,$$

that is a smoothly depending on $(x,t) \in M^n \times [0,1]$ family of diffeomorphisms

$$\{\xi_t \in Diff(M^n),\, t \in [0,1]\}.$$

The arc ξ is called *simple* if $\xi_t \in MS(M^n)$ for every $t \in ([0,1] \setminus B)$, where B is a finite set and for $t \in B$ diffeomorphisms undergo bifurcations of the following types: saddle-node, doubling period, heteroclinic tangency (see section 3 for details).

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As follows from the papers by Sh. Matsumoto [6] and P. Blanchar [1], any oriented closed surface admits isotopic Morse-Smale diffeomorphisms, who can not be connected by a simple arc. In the paper by V. Grines and O. Pochinka [3] necessary and sufficient conditions were found for the fact that the Morse-Smale diffeomorphism without heteroclinic intersections on the 3-sphere is connected by a simple arc with the "source-sink" diffeomorphism. They also constructed examples of Morse-Smale diffeomorphisms on the 3-sphere that are not joined by a simple arc due to the wild embeddings of all saddle separatrices for one of them.

In the present paper we consider a $f \in MS(M^n)$ which defined on a multidimensional not simply connected manifold M^n for $n \geq 3$. Denote by $MS_0(M^n)$ the class of homotopic to identity Morse-Smale diffeomorphisms. Let $f \in MS_0(M^n)$. Through $\mathcal{O}_x$ we denote the orbit of the point $x \in M^n$ under the diffeomorphism f. Let $\gamma \in H_1(M^n)$.

Following to [6], we say that a periodic orbit $\mathcal{O}_p$ *is homologically γ-related* to a periodic orbit $\mathcal{O}_q$ if there is a curve $c \subset M^n$ such that $\partial c = \{q\} - \{p\}$ and for some integer N such that $f^N(p) = p$ and $f^N(q) = q$, $[f^N(c) - c] = N\gamma$. The definition independents on the choice $c, N, p \in \mathcal{O}_p, q \in \mathcal{O}_q$. We say that f is *trivial* if all periodic orbits of the diffeomorphism f are 0-related otherwise f is *nontrivial*.

In section 2 isotopic diffeomorphisms $f_0, f_1 \in MS_0(\mathbb{S}^{n-1} \times \mathbb{S}^1), n \geq 3$ will constructed, one of which is trivial, the other is nontrivial. The main result of this paper is the following theorem.

Theorem. *There is no simple arc connecting a trivial diffeomorphism with a nontrivial diffeomorphism from the class $MS_0(M^n)$.*

2. The construction of a trivial-nontrivial pair of isotopic diffeomorphisms

Let
$$\mathbb{S}^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : {x_1}^2 + {x_2}^2 + \cdots + {x_{n+1}}^2 = 1\}.$$
For the sphere $\mathbb{S}^1$ also consider its complex form
$$\mathbb{S}^1 = \{e^{i2\pi\beta}, \beta \in [0;1]\}.$$
Define a diffeomorphism $\phi\colon [0;1] \to [0;1]$ by the formula:
$$\phi(\beta) = \beta + \beta\,(\beta - 1)\left(\beta - \frac{1}{2}\right).$$
Dynamics of a diffeomorphism of the circle sending a point $e^{i2\pi\beta}$ to the point $e^{i2\pi\phi(\beta)}$ is shown in Figure 1. Notice that the diffeomorphism ϕ is isotopic to the identity since there is an isotopy $\phi_t\colon [0;1] \to [0;1]$ given by the formula:
$$\phi_t(\beta) = \beta + t\beta(\beta - 1)\left(\beta - \frac{1}{2}\right),$$

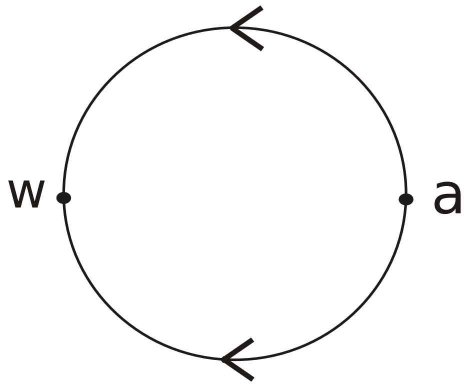

Fig. 1. Source-sink on the circle

for which $\phi_0 = id$ and $\phi_1 = g$.

For $n > 2$ let us define a diffeomorphism $\psi\colon \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ by the formula:

$$\psi(x_1, x_2, \ldots, x_n) = \left(\frac{4x_1}{5-3x_n}, \frac{4x_2}{5-3x_n}, \ldots, \frac{5x_n - 3}{5-3x_n}\right).$$

Figure 2 depicts the dynamics of such a diffeomorphism for $n = 3$. The diffeomorphism

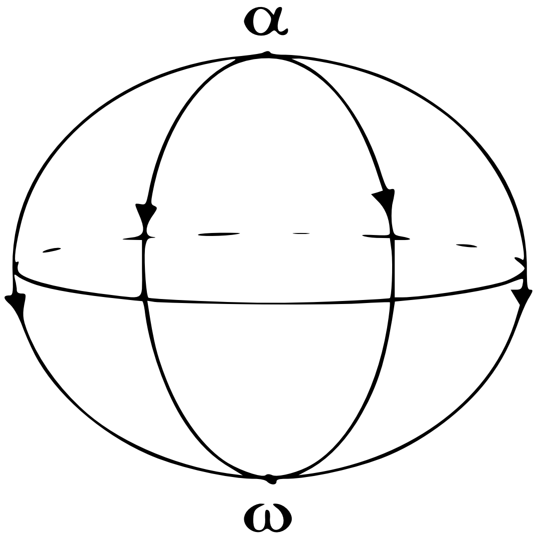

Fig. 2. Source-sink on the sphere

ψ is also isotopic to the identity since there exists an isotopy $\psi_t\colon \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ given by the formula:

$$\psi_t(x_1, x_2, \ldots, x_n) = \left(\frac{x_1(1+3t)}{t(4-3x_n)+1}, \frac{x_2(1+3t)}{t(4-3x_n)+1}, \ldots, \frac{t(4-3)+x_n}{t(4-3x_n)+1}\right),$$

for which $\psi_0 = id$ and $\psi_1 = \psi$.

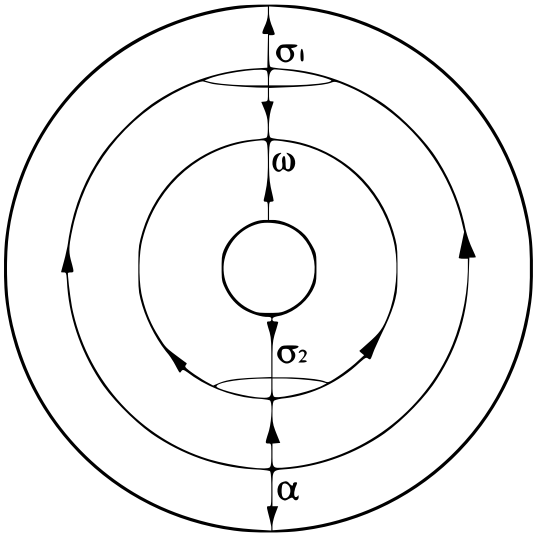

Fig. 3. Dynamic of the trivial diffeomorphism $f_0\colon \mathbb{S}^2\times\mathbb{S}^1\to\mathbb{S}^2\times\mathbb{S}^1$

Let us consider the Cartesian product of our spheres $\mathbb{S}^{n-1}\times\mathbb{S}^1$ and define a diffeomorphism $f_0\colon \mathbb{S}^{n-1}\times\mathbb{S}^1\to\mathbb{S}^{n-1}\times\mathbb{S}^1$ by the formula:

$$f_0(x_1,\dots,x_n,e^{i2\pi\beta})=(\psi(x_1,\dots,x_n),e^{i2\pi\phi(\beta)}).$$

By construction, the diffeomorphism f_0 is a Morse-Smale diffeomorphism and its non-wandering set consists of one sink, one source, and two saddle points whose invariant manifolds do not intersect. Figure 3 shows a phase portrait for the case $n=3$. Since there is an isotopy

$$f_{0,t}(x_1,\dots,x_n,e^{i2\pi\beta})=(\psi_t(x_1,\dots,x_n),e^{i2\pi\phi_t(\beta)})$$

such that $f_{0,0}=id$ and $f_{0,1}=f_0$, hence the diffeomorphism f_0 is isotopic to the identity. In addition, it is easy to see that all its fixed points are trivially related.

On the sphere $\mathbb{S}^{n-1}$ let us consider a subset of points $(x_1,\dots,x_n)$, for which $x_n\in[0,\frac{3}{5}]$ (see Figure 4 for the case $n=3$). It is diffeomorphic to n-dimensional annulus, denote it by $\mathbb{L}$. In the cartesian product $\mathbb{S}^{n-1}\times\mathbb{S}^1$ we obtain a subset $\mathbb{K}=\mathbb{L}\times\mathbb{S}^1, x_n\in[0,\frac{3}{5}]$. We define a diffeomorphism $\varphi\colon \mathbb{S}^{n-1}\times\mathbb{S}^1\to\mathbb{S}^{n-1}\times\mathbb{S}^1$ which is the identity outside $\mathbb{K}$ and on $\mathbb{K}$ is given by the formula:

$$\varphi(x_1,\dots,x_n,e^{i2\pi\beta})=(x_1,\dots,x_n,e^{i2\pi(\beta+\frac{5}{3}x_n)}).$$

We show that the diffeomorphism φ is isotopic to the identity. To do this, we construct the isotopy φ_t as follows:

1) $\varphi_t=id$ on the set $\mathbb{K}^-=\{x_1,\dots,x_n,e^{i2\pi\beta})\in\mathbb{S}^{n-1}\times\mathbb{S}^1 : x_n<0\}$;

2) $\varphi_t(x_1,\dots,x_n,e^{i2\pi\beta})(x_1,\dots,x_n,e^{i2\pi(\beta+\frac{5}{3}x_nt)})$ on the set $\mathbb{K}$;

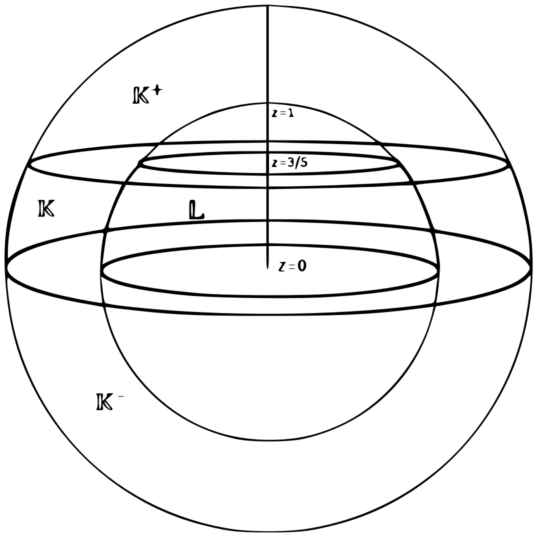

Fig. 4. The parts of $\mathbb{S}^2 \times \mathbb{S}^1$

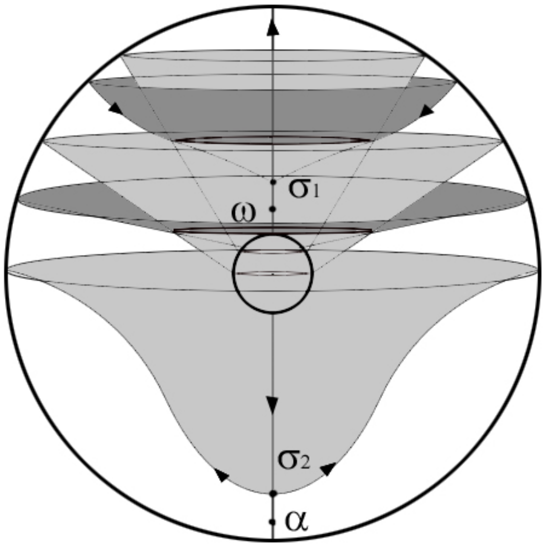

Fig. 5. Dynamic of the nontrivial diffeomorphism $f_0\colon \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^2 \times \mathbb{S}^1$

3) $\varphi_t(x_1, \ldots, x_n, e^{i2\pi\beta}) = (x_1, \ldots, x_n, e^{i2\pi(\beta+t)})$ on the set $\mathbb{K}^+ = \{(x_1, \ldots, x_n, e^{i2\pi\beta}) \in \mathbb{S}^{n-1} \times \mathbb{S}^1 : x_n \in [\frac{3}{5}, 1)\}$.

From the construction of isotopy φ_t it's clear that $\varphi_0 = id, \varphi_1 = \varphi$. We define a diffeomorphism $f_1\colon \mathbb{S}^{n-1} \times \mathbb{S}^1 \to \mathbb{S}^{n-1} \times \mathbb{S}^1$ formula

$$f_1 = \varphi f_0.$$

Then the diffeomorphism f_1 is isotopic to the identity by means of the isotopy $f_{1,t} = \varphi_t f_{0,t}$. By construction, the diffeomorphism f_1 is a Morse-Smale diffeomorphism, its non-wandering set consists of one sink, one source, and two saddle points whose two-dimensional manifolds intersect along a countable set of compact heteroclinic curves. In addition, the saddle points are not trivially related. In Figure 5 we show the Morse-Smale diffeomorphism f_1 for the case $n = 3$.

3. Simple arcs

Let us consider a smooth map $\xi\colon M^n \times [0,1] \to M^n$ — *a* smooth arc such that $\xi_t \in MS(M^n)$ for every $t \in ([0,1] \setminus B)$, where B is a finite set. For a generic set of such arcs, the diffeomorphism ξ_b, $b \in B$ has the finite non-wandering set, has no cycles and under the direction of motion along the arc, undergoes bifurcations of the following types: saddle-node, doubling period, heteroclinic tangency, for exact details see, for example, [7]. Below we give an information about these bifurcations, for exact details see, for example, [5].

Let p be a fixed point of a diffeomorphism $f\colon M^n \to M^n$. Differential Df_p induces a decomposition of the tangent space T_pM^n into a direct sum of invariant subspaces

$$T_pM^n = E^u \oplus E^c \oplus E^s.$$

Linear maps $Df_p|_{E^u}$, $Df_p|_{E^c}$, $Df_p|_{E^s}$ have eigenvalues, respectively, outside, on the boundary, inside the unit disc. There exists a unique smooth invariant submanifold W^u_p (W^s_p) of the manifold M^n tangent to E^u (E^s) at the point p and possessing the property

$$W^u_p = \{y \in M^n : \lim_{k\to-\infty} f^k(y) = p\} \qquad (W^s_p = \{y \in M^n : \lim_{k\to+\infty} f^k(y) = p\}).$$

It is called by *unstable (stable) manifold* of the point p. In particular, if $dim\, E^c = 0$, the point p is *hyperbolic*. Otherwise, there exists a smooth invariant submanifold W^c_p of the manifold M^n tangent to E^c at the point p. It is called the *central manifold* of a nonhyperbolic fixed point. A central manifold is not unique but the maps $f|_{W^c_p}$ and $f|_{\tilde{W}^c_p}$ are topologically conjugated for any central manifolds W^c_p and $\tilde{W}^c_p$.

The central, stable and unstable manifolds of a periodic point of period k is defined as the corresponding manifolds of this point as a fixed point of the diffeomorphism f^k.

In the explanatory drawings, double arrows schematically show the directions of motion with exponential contraction and expansion, and single directions indicate the directions of motion along the central manifold of the nonhyperbolic point.

1. All periodic orbits of the diffeomorphism ξ_b are hyperbolic with an exception in a one orbit $\mathcal{O}_p$ of the period k for which all eigenvalues of $(Df^k)_p$ different from 1 by absolute values except one $\lambda = 1$. The stable and the unstable manifolds of different periodic orbits of the diffeomorphism ξ_b intersect transversely and $W^s_p \cap W^u_p = \{p\}$. The transition through ξ_b is accompanied by a confluence and further disappearance of hyperbolic periodic points of the same period. This bifurcation is called *saddle-node* (see Figure 6).

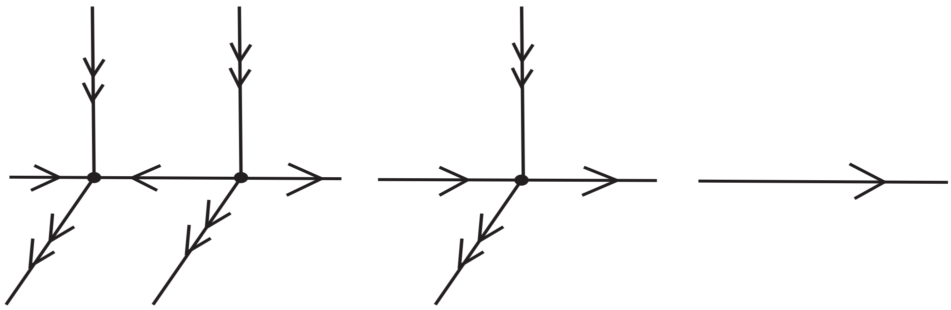

Fig. 6. Saddle-node bifurcation

2. All periodic orbits of the diffeomorphism ξ_b are hyperbolic with an exception in a one orbit $\mathcal{O}_p$ of the period k for which all eigenvalues of $(Df^k)_p$ different from 1 by the absolute value except one $\lambda = -1$. The stable and the unstable manifolds of different periodic orbits of the diffeomorphism ξ_b intersect transversely and $W^s_p \cap W^u_p = \{p\}$. By passing through ξ_b along the central manifold an attractor becomes a repeller and a periodic hyperbolic orbit of the period $2k$ is generated. Such a bifurcation is called *a doubling period* (see Figure 7).

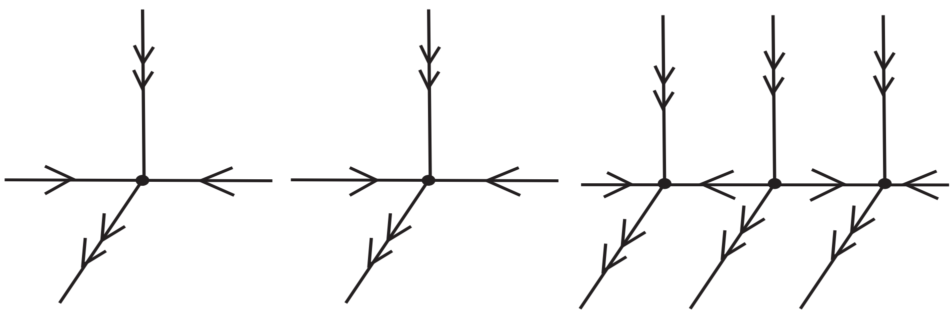

Fig. 7. Doubling period bifurcation

3. All periodic orbits of the diffeomorphism ξ_b are hyperbolic, their stable and unstable manifolds have transversal intersections everywhere except for one trajectory along which the intersection is quasi-transversal. Such a bifurcation is called a *bifurcation of a heteroclinic tangency* (see Figure 8).

4. Proof of the main result

In this section we prove Theorem. Namely, we consider two homotopic to identity Morse-Smale diffeomorphisms f_0, f_1 given on a not simply connected n-manifold M^n such that f_0 is a trivial and the diffeomorphism f_1 is a nontrivial. Let us prove that there is no simple arc joining the diffeomorphisms f_0 and f_1.

Proof. Assume the contrary: f_0 and f_1 can be joined by a simple arc. Then on this arc there are two Morse-Smale diffeomorphisms g_0 and g_1 such that:

1) g_0 and g_1 can be connected with a simple arc $\{g_t\}_{t\in[0,1]}$ with only one bifurcation point, $g_{\frac{1}{2}} = h$ (see Figure 9);

2) g_0 is a trivial diffeomorphism, g_1 — nontrivial.

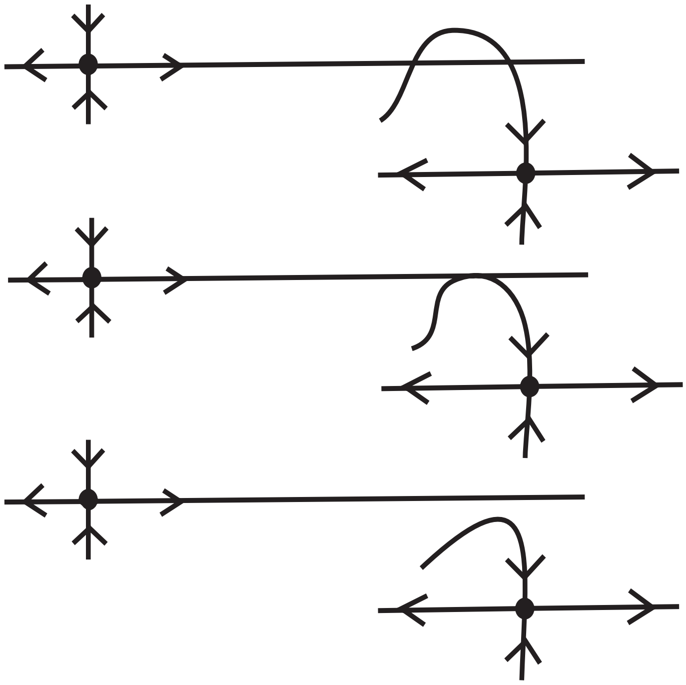

Fig. 8. Bifurcation of the heteroclinic tangency

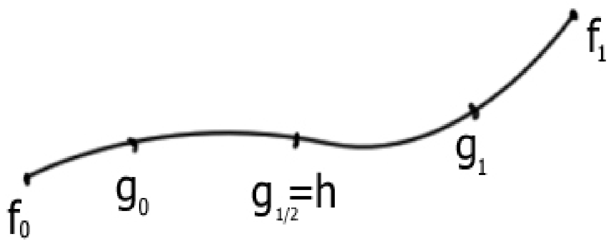

Fig. 9.

From the description of possible for a simple arc bifurcations (see the section 3) it follows that the non-wandering set Ω_h of the diffeomorphism h has a periodic orbit saddle-node $\mathcal{O}_p$ that is not trivially related to some (and, consequently, with any) other periodic orbit; in a comparison with the Morse-Smale diffeomorphism $g_t, 0 \leq t < \frac{1}{2}$, two periodic orbits of the different indexes appear for the Morse-Smale diffeomorphism $g_t, \frac{1}{2} < t < 1$.

Let $m, k \in \mathbb{N}$ be the dimensions of the unstable, stable manifolds W^u_p, W^s_p of the point p. Then $m + k = n + 1$. By an analogy with the properties of Morse-Smale diffeomorphisms (see, for example, [2] Theorem 2.1), one can establish that

$$M^n = \bigcup_{x \in \Omega_h} W^u_x = \bigcup_{x \in \Omega_h} W^s_x.$$

Since there are no cycles for the diffeomorphism h, there are hyperbolic points $q, r \in \Omega_h$ such that

$$W^u_p \cap W^s_q \neq \emptyset, \qquad W^s_p \cap W^u_r \neq \emptyset. \tag{4.1}$$

These points are not node as in this case p will 0-related with q or r by means a curve c on W^s_q or W^u_r. From the transversality of the intersection of the stable and the unstable manifolds of non-wandering points of the diffeomorphism h it follows that the invariant manifolds W^s_q and W^u_r are arbitrarily close to the point p and, therefore, are close each to other, this means that $W^s_q \cap W^u_r \neq \emptyset$.

It means that $W^s_q \cap W^u_r \neq \emptyset$ for every diffeomorphism g_t for t near $\frac{1}{2}$. Moreover, if $t < \frac{1}{2}$ then q and r are the nearest saddle points, that is the intersection $W^s_q \cap W^u_r \neq \emptyset$ consists of a finite number connected components, denote it N. Then there is compact fundamental domains F_s of $f|_{W^s_q \setminus q}$ containing exactly N connected components in the intersection with W^u_r. For $t = \frac{1}{2}$ the conditions (4.1) imply that there is a compact subset C_u of $W^u_r \setminus r$ such that the number of the connected components in the intersection $F_s \cap C_u$ is greater than $N + 2$. Due to transversality condition for g_t, t sufficiently near $\frac{1}{2}$ the number of the connected components in the intersection $F_s \cap C_u$ is preserving, that is contradicting the definition of N. □

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