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Sufficient conditions for the asymptotic stability of solutions to one class of linear systems of neutral type with periodic coefficients

T. K. Yskak

Novosibirsk State University, Novosibirsk 630090, Russia.

E-mail: istima92@mail.ru

Abstracts. In the paper we consider one class of systems of linear delay differential equations with a parameter and periodic coefficients. They belong to systems of neutral type. We prove an analogue of M. G. Krein's theorem for this class. Using a Lyapunov–Krasovskii functional, we obtain sufficient conditions for the asymptotic stability of the zero solution and establish estimates characterizing the exponential decay of solutions to the systems at infinity.

Keywords: delay differential equations, neutral type, periodic coefficients, asymptotic stability, Lyapunov–Krasovskii functional

1. Introduction

In the present paper we consider a system of linear delay differential equations of neutral type

$$\frac{d}{dt}\big(y(t) + Dy(t-\tau)\big) = \mu A(t)y(t) + B(t)y(t-\tau), \quad t > 0, \tag{1.1}$$

where D is an $(n\times n)$-matrix with constant complex entries, $A(t)$ and $B(t)$ are $(n\times n)$-matrices with continuous T-periodic complex-valued entries, $\mu > 0$ is a parameter. We assume that the spectrum of the matrix $A(t)$ belongs to the left half-plane $\mathbb{C}_- = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}$ for all $t \in [0, T]$.

Our aim is the study of the asymptotic stability of the zero solution to this system for $\mu \gg 1$.

Note that in the literature there exists a number of results for systems of the form (1.1) in the cases $D = 0$, $B(t) \equiv 0$ (ordinary differential equations [2, p. 209]) and $D = 0$, $B(t) \not\equiv 0$ (delay differential equations [6]). We formulate some of them.

At first consider the following system of ordinary differential equations with a parameter $\mu > 0$

$$\frac{d}{dt}y = \mu A(t)y, \qquad t > 0, \tag{1.2}$$

where $A(t)$ is a matrix with continuous T-periodic entries. A very important theorem of M. G. Krein holds [2, p. 209].

Theorem 1. *If the spectrum of the matrix $A(t)$ belongs to the left-half plane $\mathbb{C}_-$ for all $t \in [0, T]$, then the zero solution to (1.2) is asymptotically stable for all sufficiently large positive μ.*

Emphasize that in this theorem the asymptotic stability of the solution to (1.2) is guaranteed only for sufficiently large $\mu > 0$. This fact is essential, because there exist examples of systems of the form (1.2), the zero solution of which is unstable for arbitrary $\mu > 0$. Indeed, consider the following system of ordinary differential equations proposed by R.E. Vinograd (see, for example, [1, p. 124])

$$\frac{d}{dt}y = A(t)y, \qquad t \geq 0,$$

where

$$A(t) = \begin{pmatrix} -\frac{11}{2} + \frac{15}{2}\cos 12t & -6 + \frac{15}{2}\sin 12t \\ 6 + \frac{15}{2}\sin 12t & -\frac{11}{2} - \frac{15}{2}\cos 12t \end{pmatrix}.$$

It is easy to see that $A(t) \equiv A\left(t + \frac{\pi}{6}\right)$. Eigenvalues of the matrix $A(t)$ are constant and equal to -1 and -10, solutions of the system are given by the formula

$$y(t) = c_1 e^{2t} \begin{pmatrix} \cos 6t \\ \sin 6t \end{pmatrix} + c_2 e^{-13t} \begin{pmatrix} \sin 6t \\ -\cos 6t \end{pmatrix}.$$

From the formula it is easy to see that the zero solution to this system is unstable.

Note that a "threshold" value μ_0 was obtained in [6], beginning from which the asymptotic stability of the zero solution to (1.2) is guaranteed. To formulate the corresponding statement we need to give some notations. According to Lyapunov's criterion, the Lyapunov matrix equation

$$VA + A^*V = -I$$

has a unique solution $V = V^* > 0$ if and only if the spectrum of A belongs to the left half-plane $\mathbb{C}_-$. Here A^* is the conjugate transpose of A. Since the spectrum of the matrix $A(t)$ belongs to the left half-plane $\mathbb{C}_-$ for all $t \in [0, T]$, then by the Lyapunov criterion there exists a unique solution $\widehat{H}(t) = \widehat{H}^*(t) > 0$ to the matrix equation

$$\widehat{H}A(t) + A^*(t)\widehat{H} = -I$$

for every fixed $t \in [0, T]$. Introduce the following notations

$$H_{\max} = \max_{t\in[0,T]} \|\widehat{H}(t)\|, \qquad \nu_{\max} = \max_{t\in[0,T]} \nu(\widehat{H}(t)),$$

where $\nu(\widehat{H}(t))$ is the condition number of the matrix $\widehat{H}(t)$, $\|\widehat{H}(t)\|$ is its spectral norm.

The following theorem holds [6].

Theorem 2. *Let the spectrum of the matrix $A(t)$ belong to the left half-plane $\mathbb{C}_-$ for all $t \in [0, T]$, let N be a number such that the following inequality holds*

$$\max_{|t-s|\leq \frac{T}{N}} \|A(t) - A(s)\| \leq \frac{1}{4H_{\max}\sqrt{\nu_{\max}}}, \tag{1.3}$$

and

$$\mu_0 = \frac{2NH_{\max}}{T} \ln \nu_{\max}. \tag{1.4}$$

Then the zero solution to (1.2) *is asymptotically stable for all $\mu > \mu_0$.*

Note that the existence of N, for which (1.3) holds, is assured by the continuity of the entries of the matrices $A(t)$, $\widehat{H}(t)$ and the function $\nu(\widehat{H}(t))$. In the proof of this theorem it was essentially used the following criterion of the asymptotic stability [4] for the system

$$\frac{d}{dt}y = A(t)y, \tag{1.5}$$

where $A(t)$ is a continuous T-periodic $(n \times n)$-matrix.

Theorem 3. *I. If the zero solution to* (1.5) *is asymptotically stable then, for every continuous matrix $C(t)$ on $[0,T]$, there exists a unique solution $L(t)$ to the boundary value problem*

$$\begin{cases} \dfrac{d}{dt}L + LA(t) + A^*(t)L = -C(t), \quad 0 < t < T, \\ L(0) = L(T), \end{cases} \tag{1.6}$$

moreover, if

$$C(t) = C^*(t) > 0, \quad t \in [0,T], \tag{1.7}$$

then

$$L(t) = L^*(t) > 0, \quad t \in [0,T].$$

II. Let the right-hand side $C(t)$ be continuous on $[0,T]$ and satisfy the conditions (1.7)*. If the boundary value problem* (1.6) *has a Hermitian solution $L(t)$ such that $L(0) > 0$ then the zero solution to* (1.5) *is asymptotically stable.*

For $\mu > \mu_0$, due to the criterion of the asymptotic stability, the boundary value problem for the Lyapunov differential equation

$$\begin{cases} \dfrac{d}{dt}L + \mu LA(t) + \mu A^*(t)L = -M, \quad 0 < t < T, \\ L(0) = L(T) > 0, \end{cases} \tag{1.8}$$

with $M = M^* > 0$ has a unique solution $L(t) = L^*(t) > 0$. Moreover, as was shown in [6], the estimate holds

$$\|L(t)\| \leq \frac{2\|M\|H_{\max}(\nu_{\max})^N}{\mu}\left(1 - (\nu_{\max})^N \exp\left(-\frac{\mu T}{2H_{\max}}\right)\right)^{-1}. \tag{1.9}$$

Hereafter, we will use the same notation for T-periodic extension of the matrix $L(t)$ for $t > 0$.

In the case of $D = 0$, $B(t) \not\equiv 0$ the zero solution to (1.1) is also asymptotically stable for all sufficiently large $\mu > 0$. The authors of [6] found a "threshold" value μ_1 such that the asymptotic stability of the zero solution to (1.1) is guaranteed for $\mu > \mu_1$ We now formulate this result.

Theorem 4. *Let $D = 0$, let N be a number such that* (1.3) *holds, and let $\mu_1 > \mu_0$ satisfy the equation*

$$\mu\left(1 - (\nu_{\max})^N e^{\left(-\frac{\mu T}{2H_{\max}}\right)}\right) = 4e^{\tau/2}H_{\max}(\nu_{\max})^N \max_{\xi\in[0,T]}\|B(\xi)\|.$$

Then the zero solution to (1.1) *is asymptotically stable for all $\mu > \mu_1$.*

In the proof of this statement the authors used a modified Lyapunov–Krasovskii functional of the form

$$V(t,y) = \langle L(t)y(t), y(t)\rangle + \frac{1}{2}\int\limits_{t-\tau}^{t} e^{-(t-s)}\|y(s)\|^2\,ds,$$

where $L(t) = L^*(t) > 0$ is T-periodic extension of the solution to the boundary value problem (1.8) with $M = I$. Here the condition on μ is also essential, because, based on an example constructed by R. E. Vinograd, we can construct an example of a system of the form (1.1) with $D = 0$, $B(t) \not\equiv 0$, for which the zero solution is unstable for $\mu = 1$. Namely, consider the following system

$$\frac{d}{dt}y(t) = A(t)y(t) + B(t)y\left(t - \frac{\pi}{3}\right), \quad t > 0,$$

where

$$A(t) = \begin{pmatrix} -\frac{11}{4} + \frac{15}{4}\cos 12t & -3 + \frac{15}{4}\sin 12t \\ \\ 3 + \frac{15}{4}\sin 12t & -\frac{11}{4} - \frac{15}{4}\cos 12t \end{pmatrix},$$

$$B(t) = \frac{e^{2\pi/3}}{2}\begin{pmatrix} -\frac{11}{2} + \frac{15}{2}\cos 12t & -6 + \frac{15}{2}\sin 12t \\ \\ 6 + \frac{15}{2}\sin 12t & -\frac{11}{2} - \frac{15}{2}\cos 12t \end{pmatrix}.$$

Using the example described above, we obtain that the vector-function

$$y(t) = ce^{2t}\begin{pmatrix} \cos 6t \\ \sin 6t \end{pmatrix}$$

is a solution to this system. Therefore, its zero solution is unstable, though the eigenvalues of the matrix $A(t)$ are constant and equal to $-\frac{1}{2}$ and -5.

It should be noted that Theorem 4 is an analogue of M. G. Krein's result formulated in Theorem 1 for systems of linear delay differential equations of the form (1.1) with $D = 0$. Our aim is to establish such analogue in the case of $D \neq 0$. In Section 2 we formulate sufficient conditions for the asymptotic stability of the zero solution to (1.1) and obtain estimates of solutions to (1.1) with $\mu = 1$. In Section 3 we give conditions on the matrix D and a "threshold" value on the parameter μ, beginning from which the zero solution to (1.1) is asymptotically stable.

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2. Asymptotic stability of the zero solution to the system of neutral type

Consider the initial value problem for (1.1) with $\mu = 1$

$$\begin{cases} \dfrac{d}{dt}\big(y(t) + Dy(t-\tau)\big) = A(t)y(t) + B(t)y(t-\tau), \quad t > 0, \\ y(t) = \varphi(t), \quad t \in [-\tau, 0], \\ y(+0) = \varphi(0), \end{cases} \tag{2.1}$$

where D is an $(n \times n)$-matrix with constant entries, $A(t)$ and $B(t)$ are $(n \times n)$-matrices with continuous T-periodic entries. We assume that the spectrum of $A(t)$ belongs to the left-half plane $\mathbb{C}_-$ for all $t \in [0, T]$. When studying the asymptotic stability of the zero solution, we use the following modified Lyapunov–Krasovskii functional introduced in [3, 5]

$$V(t,y) = \langle H(t)(y(t) + Dy(t-\tau)), (y(t) + Dy(t-\tau))\rangle + \int\limits_{t-\tau}^{t} \langle K(t-s)y(s), y(s)\rangle \, ds. \qquad (2.2)$$

We now formulate a theorem which is an analogue of statements from [3, 5].

Theorem 5. *Let there exist a smooth T-periodic matrix $H(t) = H^*(t)$ such that $H(t) > 0$ and a matrix $K(s) = K^*(s) \in C^1[0,\tau]$ such that*

$$K(s) > 0, \quad \frac{d}{ds}K(s) < 0, \quad s \in [0,\tau].$$

Denote by $q(t)$ the minimal eigenvalue of the matrix

$$Q(t) = H^{-\frac{1}{2}}(t)\Big(Q_{11}(t) - Q_{12}(t)Q_{22}^{-1}Q_{12}^*(t)\Big)H^{-\frac{1}{2}}(t),$$

where

$$Q_{11}(t) = -\frac{d}{dt}H(t) - H(t)A(t) - A^*(t)H(t) - K(0),$$
$$Q_{12}(t) = H(t)A(t)D + K(0)D - H(t)B(t),$$
$$Q_{22} = K(\tau) - D^*K(0)D > 0.$$

Choose a number $k > 0$ such that

$$\frac{d}{ds}K(s) + kK(s) \le 0, \quad s \in [0,\tau].$$

Then for the solution to the problem (2.1) the following estimate holds

$$\langle H(t)(y(t)+Dy(t-\tau)), (y(t)+Dy(t-\tau))\rangle + \int\limits_{t-\tau}^{t} \langle K(t-s)y(s), y(s)\rangle \, ds \le \exp\left(-\int_0^t \gamma(s)ds\right)$$

$$\times \left(\langle H(0)(\varphi(0) + D\varphi(-\tau)), (\varphi(0) + D\varphi(-\tau))\rangle + \int\limits_{-\tau}^{0} \langle K(-s)\varphi(s), \varphi(s)\rangle \, ds\right), \qquad (2.3)$$

where

$$\gamma(t) = \min\{q(t), k\}.$$

Proof. Let $y(t)$ be a solution to the initial value problem (2.1). Consider the modified Lyapunov–Krasovskii functional (2.2) on the solution $y(t)$. Its derivative has the form

$$\frac{d}{dt}V(t,y) = \left\langle \left(\frac{d}{dt}H(t) + H(t)A(t) + A^*(t)H(t) + K(0)\right) y(t), y(t) \right\rangle$$

$$+\left\langle\left(\frac{d}{dt}H(t)D+A^*(t)H(t)D+H(t)B(t)\right)y(t-\tau),y(t)\right\rangle$$

$$+\left\langle y(t),\left(\frac{d}{dt}H(t)D+A^*(t)H(t)D+H(t)B(t)\right)y(t-\tau)\right\rangle$$

$$+\left\langle\left(D^*\frac{d}{dt}H(t)D+D^*H(t)B(t)+B^*(t)H(t)D-K(\tau)\right)y(t-\tau),y(t-\tau)\right\rangle$$

$$+\int\limits_{t-\tau}^{t}\left\langle\frac{d}{dt}K(t-s)y(s),y(s)\right\rangle ds.$$

Denote

$$C(t)=-\begin{pmatrix} C_{11}(t) & C_{12}(t)\\ C_{12}^*(t) & C_{22}(t)\end{pmatrix},$$

where

$$C_{11}(t)=\frac{d}{dt}H(t)+H(t)A(t)+A^*(t)H(t)+K(0),$$

$$C_{12}(t)=\frac{d}{dt}H(t)D+A^*(t)H(t)D+H(t)B(t),$$

$$C_{22}(t)=D^*\frac{d}{dt}H(t)D+D^*H(t)B(t)+B^*(t)H(t)D-K(\tau).$$

Then we have

$$\frac{d}{dt}V(t,y)+\left\langle C(t)\begin{pmatrix} y(t)\\ y(t-\tau)\end{pmatrix},\begin{pmatrix} y(t)\\ y(t-\tau)\end{pmatrix}\right\rangle$$

$$-\int\limits_{t-\tau}^{t}\left\langle\frac{d}{dt}K(t-s)y(s),y(s)\right\rangle ds=0. \qquad (2.4)$$

It is easy to verify the validity of the following identity

$$\left\langle C(t)\begin{pmatrix} z_1\\ z_2\end{pmatrix},\begin{pmatrix} z_1\\ z_2\end{pmatrix}\right\rangle\equiv\left\langle\begin{pmatrix} Q_{11}(t) & Q_{12}(t)\\ Q_{12}^*(t) & Q_{22}\end{pmatrix}\begin{pmatrix} z_1+Dz_2\\ z_2\end{pmatrix},\begin{pmatrix} z_1+Dz_2\\ z_2\end{pmatrix}\right\rangle.$$

Indeed,

$$\left\langle C(t)\begin{pmatrix} z_1\\ z_2\end{pmatrix},\begin{pmatrix} z_1\\ z_2\end{pmatrix}\right\rangle$$

$$\equiv\left\langle\begin{pmatrix} C_{11}(t) & C_{12}(t)\\ C_{12}^*(t) & C_{22}(t)\end{pmatrix}\begin{pmatrix} I & -D\\ 0 & I\end{pmatrix}\begin{pmatrix} z_1+Dz_2\\ z_2\end{pmatrix},\begin{pmatrix} I & -D\\ 0 & I\end{pmatrix}\begin{pmatrix} z_1+Dz_2\\ z_2\end{pmatrix}\right\rangle.$$

Hence,

$$\left\langle C(t)\begin{pmatrix} z_1\\ z_2\end{pmatrix},\begin{pmatrix} z_1\\ z_2\end{pmatrix}\right\rangle$$

$$\equiv\left\langle\begin{pmatrix} I & 0\\ -D^* & I\end{pmatrix}\begin{pmatrix} C_{11}(t) & C_{12}(t)\\ C_{12}^*(t) & C_{22}(t)\end{pmatrix}\begin{pmatrix} I & -D\\ 0 & I\end{pmatrix}\begin{pmatrix} z_1+Dz_2\\ z_2\end{pmatrix},\begin{pmatrix} z_1+Dz_2\\ z_2\end{pmatrix}\right\rangle.$$

Multiplying the matrices, we obtain the required identity. By the conditions of the theorem, $Q_{22} > 0$. Hence, the following representation holds

$$\left\langle \begin{pmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{12}^*(t) & Q_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\rangle = \langle (Q_{11}(t) - Q_{12}(t)Q_{22}^{-1}Q_{12}^*(t))z_1, z_1 \rangle$$

$$+\langle Q_{22}^{-1}(Q_{22}z_2 + Q_{12}^*(t)z_1), (Q_{22}z_2 + Q_{12}^*(t)z_1) \rangle.$$

Consequently, from (2.4) we obtain

$$\frac{d}{dt}V(t,y) \le -\langle [Q_{11}(t) - Q_{12}(t)Q_{22}^{-1}Q_{12}^*(t)](y(t) + Dy(t-\tau)), (y(t) + Dy(t-\tau)) \rangle$$

$$+\int_{t-\tau}^{t} \left\langle \frac{d}{dt}K(t-s)y(s), y(s) \right\rangle ds$$

or

$$\frac{d}{dt}V(t,y) \le -\langle Q(t)H^{\frac{1}{2}}(t)(y(t) + Dy(t-\tau), H^{\frac{1}{2}}(t)(y(t) + Dy(t-\tau)) \rangle$$

$$+\int_{t-\tau}^{t} \left\langle \frac{d}{dt}K(t-s)y(s), y(s) \right\rangle ds.$$

Then, by the definitions of $q(t)$, k, and $\gamma(t)$, we have

$$\frac{d}{dt}V(t,y) + \gamma(t)V(t,y) \le 0.$$

The estimate (2.3) immediately follows from here.

Theorem is proved. □

Remark 1. It is easy to see that $\gamma(t)$ is a T-periodic function.

Remark 2. Generally speaking, the asymptotic stability does not follow from Theorem 5 because the conditions of the theorem do not guarantee that $\exp\left(-\int_0^t \gamma(s)\,ds\right) \to 0$ as $t \to \infty$.

Hereafter, we assume that the conditions of Theorem 5 hold. Note that from the conditions on the matrix $K(s)$ and Lyapunov's criterion it follows that the spectrum of the matrix D belongs to the unit disk $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$. Indeed, since $Q_{22} > 0$ then

$$K(0) - D^*K(0)D = P > 0;$$

i.e., the Hermitian positive definite matrix $K(0)$ satisfies the Lyapunov discrete equation with a Hermitian positive definite right-hand side. According to Lyapunov's criterion, this equation has a unique Hermitian positive definite solution if and only if the spectrum of D belongs to the unit disk.

Lemma 1. *Let $x(t)$ be a T-periodic function. Then*

$$\int\limits_0^t x(s)ds \geq \frac{t}{T}S + q,$$

where

$$S = \int\limits_0^T x(s)ds, \quad q = \min_{\xi\in[0,T]} \left(\int\limits_0^\xi x(s)ds - \frac{\xi}{T}\int\limits_0^T x(s)ds \right).$$

Proof. Let $t \in [nT, (n+1)T]$. Then,

$$\int\limits_0^t x(s)ds = \int\limits_0^{nT} x(s)ds + \int\limits_{nT}^t x(s)ds.$$

Because of T-periodicity of the function $x(t)$ we have

$$\int\limits_0^t x(s)ds = nS + \int\limits_0^{t-nT} x(s)ds.$$

This is equivalent to the equality

$$\int\limits_0^t x(s)ds = \frac{t}{T}S + \int\limits_0^{t-nT} x(s)ds - \frac{t-nT}{T}S.$$

From here the statement of Lemma 1 follows.

Lemma is proved. □

Denote

$$\Phi = \max_{t\in[-\tau,0]} \|\varphi(t)\|,$$

$$\hat{\kappa} = \max_{\xi\in[0,T]} \|H^{-1}(\xi)\|^{1/2} \left(2\|H(0)\|(1+\|D\|^2) + \int\limits_0^\tau \|K(s)\|\, ds \right)^{\frac{1}{2}}, \tag{2.5}$$

$$\square = \exp\left(\max_{\xi\in[0,T]} \left(\frac{\xi}{T}\int\limits_0^T \frac{\gamma(s)}{2}ds - \int\limits_0^\xi \frac{\gamma(s)}{2}ds \right)\right).$$

In the next theorem we establish estimates of solutions to the initial value problem (2.1), which are similar to estimates from [3, 5].

Theorem 6. *Suppose that the conditions of Theorem 5 hold. Let l be the minimal positive integer such that $\|D^l\| < 1$ and let $\Delta = \frac{1}{T}\int_0^T \frac{\gamma(s)}{2}ds > 0$.*

1. If $\|D^l\|\exp(\Delta l\tau) < 1$*, then for the solution to the problem* (2.1) *the estimate holds*

$$\|y(t)\| \leq \Phi\left(\hat{\kappa}\Box(1-\|D^l\|e^{\Delta l\tau})^{-1}\sum_{j=0}^{l-1}\|D^j\|e^{\Delta j\tau} + \max\{\|D\|e^{\Delta\tau},\dots,\|D^l\|e^{\Delta l\tau}\}\right)e^{-\Delta t}.$$

2. If $\|D^l\|\exp(\Delta l\tau) = 1$*, then for the solution to the problem* (2.1) *the estimate holds*

$$\|y(t)\| \leq \Phi\left(\hat{\kappa}\Box\left(1+\frac{t}{l\tau}\right)\sum_{j=0}^{l-1}\|D^j\|e^{\Delta j\tau} + \max\{\|D\|e^{\Delta\tau},\dots,\|D^{l-1}\|e^{(l-1)\Delta\tau},1\}\right)e^{-\Delta t}.$$

3. If $\|D^l\|\exp(\Delta l\tau) > 1$*, then for the solution to the problem* (2.1) *the estimate holds*

$$\|y(t)\| \leq \Phi\Bigg(\hat{\kappa}\Box\|D^l\|^{-1}(1-\|D^l\|^{-1}e^{-\Delta l\tau})^{-1}\sum_{j=0}^{l-1}\|D^j\|e^{\Delta j\tau}$$

$$+\|D^l\|^{-1}\max\{1,\|D\|,\dots,\|D^{l-1}\|\}\Bigg)e^{\frac{\ln(\|D^l\|)}{l\tau}t}.$$

Proof. Since

$$\langle H(0)(\varphi(0)+D\varphi(-\tau)),(\varphi(0)+D\varphi(-\tau))\rangle \leq \Phi^2 2\|H(0)\|(1+\|D\|^2)$$

and

$$\int_{-\tau}^{0}\langle K(-s)\varphi(s),\varphi(s)\rangle\,ds \leq \Phi^2\int_{-\tau}^{0}\|K(-s)\|\,ds,$$

then we have

$$V(0,\varphi) \leq \Phi^2\left(2\|H(0)\|(1+\|D\|^2)+\int_0^{\tau}\|K(s)\|\,ds\right).$$

From (2.3) and (2.5) for the solution to the problem (2.1) the following estimate holds

$$\langle H(t)(y(t)+Dy(t-\tau)),(y(t)+Dy(t-\tau))\rangle \leq \exp\left(-\int_0^t\gamma(s)ds\right)\Phi^2\frac{\hat{\kappa}^2}{\max\limits_{\xi\in[0,T]}\|H^{-1}(\xi)\|}.$$

Since $H(t)$ is a Hermitian positive definite matrix, then

$$\|y(t)+Dy(t-\tau)\| \leq \Phi\hat{\kappa}\exp\left(-\int_0^t\frac{\gamma(s)}{2}ds\right).$$

Taking into account the inequality

$$\|y(t)\| \leq \|y(t)+Dy(t-\tau)\|+\|Dy(t-\tau)\|,$$

we obtain

$$\|y(t)\| \le \Phi\hat{\kappa}\exp\left(-\int\limits_0^t \frac{\gamma(s)}{2}ds\right) + \|Dy(t-\tau)\|.$$

Let $t \in (k\tau, (k+1)\tau]$, $k \in \mathbb{N}$. Then, as above, we have

$$\|Dy(t-\tau)\| \le \|Dy(t-\tau) + D^2y(t-2\tau)\| + \|D^2y(t-2\tau)\|.$$

Hence,

$$\|Dy(t-\tau)\| \le \|D\|\Phi\hat{\kappa}\exp\left(-\int\limits_0^{t-\tau} \frac{\gamma(s)}{2}ds\right) + \|D^2y(t-2\tau)\|,$$

and therefore,

$$\|y(t)\| \le \Phi\hat{\kappa}\exp\left(-\int\limits_0^t \frac{\gamma(s)}{2}ds\right) + \|D\|\Phi\hat{\kappa}\exp\left(-\int\limits_0^{t-\tau} \frac{\gamma(s)}{2}ds\right) + \|D^2y(t-2\tau)\|.$$

Repeating similar reasonings, we obtain

$$\|y(t)\| \le \Phi\hat{\kappa}\sum_{j=0}^{k}\|D^j\|\exp\left(-\int\limits_0^{t-j\tau} \frac{\gamma(s)}{2}ds\right) + \|D^{k+1}y(t-(k+1)\tau)\|.$$

Using the definition of matrix norm, it is easy to show the validity of the following inequality

$$\|y(t)\| \le \Phi\left(\hat{\kappa}\sum_{j=0}^{k}\|D^j\|\exp\left(-\int\limits_0^{t-j\tau} \frac{\gamma(s)}{2}ds\right) + \|D^{k+1}\|\right).$$

In virtue of Lemma 1, the definition of Δ, and (2.5), we obtain

$$\|y(t)\| \le \Phi\left(\hat{\kappa}\Box\sum_{j=0}^{k}\|D^j\|\exp\left(-\Delta(t-j\tau)\right) + \|D^{k+1}\|\right),$$

which is equivalent to

$$\|y(t)\| \le \Phi\left(\hat{\kappa}\Box\sum_{j=0}^{k}\|D^j\|e^{\Delta j\tau} + \|D^{k+1}\|e^{\Delta t}\right)e^{-\Delta t}. \tag{2.6}$$

Case 1: $\|D^l\|e^{l\Delta\tau} < 1$. Taking into account that $t \in (k\tau, (k+1)\tau]$, from (2.6) we have

$$\|y(t)\| \le \Phi\left(\hat{\kappa}\Box\sum_{j=0}^{\infty}\|D^j\|e^{\Delta j\tau} + \|D^{k+1}\|e^{\Delta(k+1)\tau}\right)e^{-\Delta t}.$$

Then,

$$\|y(t)\| \le \Phi\left(\hat{\kappa}\Box\sum_{j=0}^{\infty}\|D^j\|e^{\Delta j\tau} + \max\{\|D\|e^{\Delta\tau}, \dots, \|D^l\|e^{\Delta l\tau}\}\right)e^{-\Delta t}.$$

Rewrite the series

$$\sum_{j=0}^{\infty}\|D^j\|e^{\Delta j\tau}=\sum_{j=0}^{l-1}\|D^j\|e^{\Delta j\tau}+\sum_{j=l}^{2l-1}\|D^j\|e^{\Delta j\tau}+\sum_{j=2l}^{3l-1}\|D^j\|e^{\Delta j\tau}+\dots$$

and estimate it

$$\sum_{j=0}^{\infty}\|D^j\|e^{\Delta j\tau}\le\sum_{j=0}^{l-1}\|D^j\|e^{\Delta j\tau}+\|D^l\|e^{\Delta l\tau}\sum_{j=0}^{l-1}\|D^j\|e^{\Delta j\tau}+(\|D^l\|e^{\Delta l\tau})^2\sum_{j=0}^{l-1}\|D^j\|e^{\Delta j\tau}+\dots.$$

Hence, we obtain the required estimate.

Case 2: $\|D^l\|e^{l\Delta\tau}=1$. Taking into account that $t\in(k\tau,(k+1)\tau]$, from (2.6) we have

$$\|y(t)\|\le\Phi\left(\hat{\kappa}\Box\sum_{j=0}^{k}\|D^j\|e^{\Delta j\tau}+\max\{\|D\|e^{\Delta\tau},\dots,\|D^{l-1}\|e^{(l-1)\Delta\tau},1\}\right)e^{-\Delta t}.$$

It is easy to see that

$$\sum_{j=0}^{k}\|D^j\|e^{\Delta j\tau}\le\left(1+\frac{t}{l\tau}\right)\sum_{j=0}^{l-1}\|D^j\|e^{\Delta j\tau}.$$

From here we immediately obtain the required estimate.

Case 3: $\|D^l\|e^{l\Delta\tau}>1$. Let $k\in(ml,(m+1)l]$. Then

$$\sum_{j=0}^{k}\|D^j\|e^{\Delta j\tau}\le\sum_{j=0}^{l-1}\|D^j\|e^{\Delta j\tau}+\|D^l\|e^{\Delta l\tau}\sum_{j=0}^{l-1}\|D^j\|e^{\Delta j\tau}+\dots$$

$$+(\|D^l\|e^{\Delta l\tau})^m\sum_{j=0}^{l-1}\|D^j\|e^{\Delta j\tau}$$

or

$$\sum_{j=0}^{k}\|D^j\|e^{\Delta j\tau}\le(\|D^l\|e^{\Delta l\tau})^m\sum_{j=0}^{l-1}\|D^j\|e^{\Delta j\tau}\big(1+(\|D^l\|e^{\Delta l\tau})^{-1}+\dots+(\|D^l\|e^{\Delta l\tau})^{-m}\big).$$

Hence,

$$\sum_{j=0}^{k}\|D^j\|e^{\Delta j\tau}\le(\|D^l\|e^{\Delta l\tau})^m(1-\|D^l\|^{-1}e^{-\Delta l\tau})^{-1}\sum_{j=0}^{l-1}\|D^j\|e^{\Delta j\tau}.$$

Since $t\in(ml\tau,(m+1)l\tau]$ and $\|D^l\|>e^{-l\Delta\tau}$, from (2.6) we have

$$\|y(t)\|\le\Phi\left(\hat{\kappa}\Box(\|D^l\|e^{\Delta l\tau})^m(1-\|D^l\|^{-1}e^{-\Delta l\tau})^{-1}\sum_{j=0}^{l-1}\|D^j\|e^{\Delta j\tau}+\|D^{k+1}\|e^{\Delta t}\right)e^{-\Delta t}.$$

Taking into account that $k\in(ml,(m+1)l]$, we obtain

$$\|D^{k+1}\|\le\|D^l\|^m\|D^{k+1-ml}\|\le\|D^l\|^m\max\{1,\|D\|,\dots,\|D^{l-1}\|\}.$$

Therefore,

$$\|y(t)\| \le \Phi\bigg(\hat\kappa\Box\|D^l\|^m e^{\Delta m l\tau}(1-\|D^l\|^{-1}e^{-\Delta l\tau})^{-1}\sum_{j=0}^{l-1}\|D^j\|e^{\Delta j\tau}$$

$$+\|D^l\|^m\max\{1,\|D\|,\dots,\|D^{l-1}\|\}e^{\Delta t}\bigg)e^{-\Delta t}.$$

Since $t\in(ml\tau,(m+1)l\tau]$, we have

$$\|y(t)\| \le \Phi\bigg(\hat\kappa\Box(1-\|D^l\|^{-1}e^{-\Delta l\tau})^{-1}\sum_{j=0}^{l-1}\|D^j\|e^{\Delta j\tau}$$

$$+\max\{1,\|D\|,\dots,\|D^{l-1}\|\}\bigg)\|D^l\|^{\frac{t}{l\tau}-1}.$$

From here the required inequality follows.

Theorem is proved. □

Remark 3. From Theorem 6 the asymptotic stability of the zero solution to (1.1) follows for $\mu=1$.

3. An analogue of M. G. Krein's theorem for the system of neutral type

Consider (1.1). We assume that the spectrum of the matrix $A(t)$ belongs to the left-half plane $\mathbb{C}_-$ for all $t\in[0,T]$, the spectrum of the matrix D lies in the unit disk $\{\lambda\in\mathbb{C}:|\lambda|<1\}$. Take $K(s)=Me^{-\beta s}$ and $H(t)$ as a solution to the Lyapunov differential equation

$$\begin{cases} \dfrac{d}{dt}H+\mu HA(t)+\mu A^*(t)H=-2M,\\ H(0)=H(T)>0,\end{cases} \tag{3.1}$$

where M is a solution to the Lyapunov discrete equation

$$M-D^*MD=I,$$

$$\beta=-\max_{\xi\in[0,T]}\frac{1}{2\tau}\ln\bigg(1-\frac{1}{2\|M\|}\Big[1-\|D\|^2\big(64\|A(\xi)\|^2\|M^{-1}\|\|M\|^2H^2_{\max}(\nu_{\max})^{2N}+\|M\|$$

$$+16\|A(\xi)\|\|M\|H_{\max}(\nu_{\max})^N\big)\Big]\bigg).$$

From Theorems 2 and 3 it follows that the boundary value problem (3.1) has a Hermitian positive definite solution for $\mu>\mu_0$, where μ_0 is defined by (1.4). Extend it T-periodically keeping the same notation.

Thereafter, we will use a well-known lemma (see, for example, [7, p. 495]).

Lemma 2. *Let P_{11}, P_{12}, and P_{22} be $(n\times n)$-matrices, then the following statements are equivalent:*

1) $P=\begin{pmatrix} P_{11} & P_{12}\\ P_{12}^* & P_{22}\end{pmatrix}>0,$ 2) $\begin{cases} P_{11}>0,\\ P_{22}-P_{12}^*P_{11}^{-1}P_{12}>0,\end{cases}$ 3) $\begin{cases} P_{22}>0,\\ P_{11}-P_{12}P_{22}^{-1}P_{12}^*>0.\end{cases}$

Theorem 7. *Let N be a number such that* (1.3) *holds and*

$$\|D\| < \min_{\xi\in[0,T]} \Big((64\|A(\xi)\|^2\|M^{-1}\|\|M\|^2 H_{\max}^2 (\nu_{\max})^{2N} + \|M\|$$

$$+16\|A(\xi)\|\|M\|H_{\max}(\nu_{\max})^N \Big)^{-\frac{1}{2}}.$$

Denote by μ_2 the maximal root of the following equation

$$128\|B(t)\|\|M^{-1}\|\|A(t)\|\|D\|\frac{(\|M\|H_{\max}(\nu_{\max})^N)^2}{\mu}$$

$$+16\|B(t)\|\|D\|\frac{\|M\|H_{\max}(\nu_{\max})^N}{\mu} + 64\|B(t)\|^2\|M^{-1}\|\left(\frac{\|M\|H_{\max}(\nu_{\max})^N}{\mu}\right)^2$$

$$= \frac{1}{2} - \frac{\|D\|^2}{2}\big(\|A(t)\|^2\|M^{-1}\|(8\|M\|H_{\max}(\nu_{\max})^N)^2 + \|M\|$$

$$+16\|A(t)\|\|M\|H_{\max}(\nu_{\max})^N\big),$$

and

$$\mu_3 = \frac{2H_{\max}(N\ln(\nu_{\max}) + \ln(2))}{T}.$$

Then the zero solution to (1.1) *is asymptotically stable for all* $\mu > \max\{\mu_2, \mu_3\}$.

Proof. In virtue of Theorems 5 and 6, it is sufficient to show that $Q(t) > 0$ for $t \in [0,T]$. By Lemma 2, this is equivalent to the inequality

$$Q_{22} - Q_{12}^*(t)Q_{11}^{-1}(t)Q_{12}(t) > 0, \quad t\in[0,T].$$

Taking into account the definitions of the matrices $Q_{11}(t)$, $Q_{12}(t)$, Q_{22} and the positive definiteness of the matrix $H(t)$, it is sufficient to establish that

$$K(\tau) - D^*K(0)D - \Big(D^*\mu A^*(t)H(t) + D^*K(0)$$

$$-B^*(t)H(t)\Big)M^{-1}\Big(H(t)\mu A(t)D + K(0)D - H(t)B(t)\Big) > 0.$$

According to the choice of the matrices $H(t)$ and $K(s)$, we can present the expression in the left-hand side as a sum

$$Me^{-\beta\tau} - D^*MD - \Big(D^*\mu A^*(t)H(t) + D^*M$$

$$-B^*(t)H(t)\Big)M^{-1}\Big(H(t)\mu A(t)D + MD - H(t)B(t)\Big) = S_1 + S_2 + S_3,$$

where

$$S_1 = M - D^*MD - \Big(D^*\mu A^*(t)H(t) + D^*M\Big)M^{-1}\Big(H(t)\mu A(t)D + MD\Big),$$

$$S_2 = -(1-e^{-\beta\tau})M,$$

$$S_3 = B^*(t)H(t)M^{-1}\Big(H(t)\mu A(t)D + MD - H(t)B(t)\Big)$$
$$+\Big(D^*\mu A^*(t)H(t) - D^*M\Big)M^{-1}H(t)B(t).$$

We now show that $S_1 > 0$, $\|S_2\| < \frac{\|S_1\|}{2}$, $\|S_3\| < \frac{\|S_1\|}{2}$.

Since $\mu > \mu_3$, from (1.9) the estimate follows

$$\|H(t)\| \leq \frac{8\|M\|H_{\max}(\nu_{\max})^N}{\mu}. \tag{3.2}$$

Show that $S_1 > 0$. Taking into account the definition of M, we have

$$\langle S_1 v, v\rangle \geq u(t)\|v\|^2,$$

where

$$u(t) = 1 - \|D\|^2\Big(\mu^2\|A(t)\|^2\|H(t)\|^2\|M^{-1}\| + \|M\| + 2\mu\|A(t)\|\|H(t)\|\Big).$$

Using the condition on D and (3.2), it is easy to verify that

$$u(t) > 0, \quad t \in [0, T].$$

Therefore $S_1 > 0$.

Prove that $\|S_2\| < \frac{\|S_1\|}{2}$. It is easy to see that

$$(1 - e^{-\beta\tau})\|M\| < \frac{u(t)}{2}.$$

Indeed, this inequality can be rewritten in the following form

$$\beta < -\frac{1}{\tau}\ln\left(1 - \frac{1 - \|D\|^2\Big(\mu^2\|A(t)\|^2\|H(t)\|^2\|M^{-1}\| + \|M\| + 2\mu\|A(t)\|\|H(t)\|\Big)}{2\|M\|}\right).$$

Taking into account the choice of β and (3.2), we obtain that this inequality is satisfied.

Finally, we prove that $\|S_3\| < \frac{\|S_1\|}{2}$. This inequality is satisfied if the following inequality is true

$$2\|B(t)\|\|M^{-1}\|\mu\|A(t)\|\|D\|\|H(t)\|^2 + 2\|B(t)\|\|D\|\|H(t)\| + \|B(t)\|^2\|M^{-1}\|\|H(t)\|^2 < \frac{u(t)}{2}.$$

Taking into account (3.2), it is sufficient to prove the estimate

$$2\|B(t)\|\|M^{-1}\|\|A(t)\|\|D\|\frac{(8\|M\|H_{\max}(\nu_{\max})^N)^2}{\mu}$$
$$+2\|B(t)\|\|D\|\frac{8\|M\|H_{\max}(\nu_{\max})^N}{\mu} + \|B(t)\|^2\|M^{-1}\|\left(\frac{8\|M\|H_{\max}(\nu_{\max})^N}{\mu}\right)^2$$
$$< \frac{1}{2} - \frac{\|D\|^2}{2}\Big(\|A(t)\|^2\|M^{-1}\|(8\|M\|H_{\max}(\nu_{\max})^N)^2 + \|M\|$$
$$+16\|A(t)\|\|M\|H_{\max}(\nu_{\max})^N\Big).$$

But this inequality is satisfied because $\mu > \mu_2$.

Theorem is proved. □

Remark 4. Note that the lower bound on μ is essential. Indeed, consider the following example, which is an analogue to R. E. Vinograd's example

$$\frac{d}{dt}\left(y(t)+\alpha y\left(t-\frac{\pi}{3}\right)\right)=A(t)y(t)+\alpha A(t)y\left(t-\frac{\pi}{3}\right), \quad t>0,$$

where α is a number,

$$A(t)=\begin{pmatrix} -\frac{11}{2}+\frac{15}{2}\cos 12t & -6+\frac{15}{2}\sin 12t \\ 6+\frac{15}{2}\sin 12t & -\frac{11}{2}-\frac{15}{2}\cos 12t \end{pmatrix}.$$

Eigenvalues of $A(t)$ are constant and equal to -1 and -10. It is easy to see that the vector-function

$$y(t)=ce^{2t}\begin{pmatrix}\cos 6t\\ \sin 6t\end{pmatrix}$$

is a solution to this system. Hence, the zero solution is unstable.

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