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On properties of solutions to one class of systems of nonlinear differential equations with parameters[1]

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Abstracts. We consider a class of systems of nonlinear ordinary differential equations with parameters. In particular, systems of such type arise when modeling the multistage synthesis of a substance. We study properties of solutions to the systems and propose a method for approximate solving the systems in the case of very large coefficients. We establish approximation estimates and show that the convergence rate depends on the parameters characterizing the nonlinearity of the systems. Moreover, the larger the coefficients of the systems, the more exact the approximate solutions. Thereby this method allows us to avoid difficulties arising inevitably when solving systems of nonlinear differential equations with very large coefficients.

Keywords: systems of ordinary differential equations, Cauchy problem, large coefficients, estimates for solutions, limit theorems.

1. Introduction

Consider the following system of ordinary differential equations

$$\begin{cases} \dfrac{dx_1}{dt} = g(t, x_n) - \dfrac{n-1}{\tau} x_1, \quad t > 0, \\ \dfrac{dx_j}{dt} = \dfrac{n-1}{\tau}(x_{j-1} - x_j), \quad j = 2, \dots, n-1, \\ \dfrac{dx_n}{dt} = \dfrac{n-1}{\tau} x_{n-1} - \theta x_n. \end{cases} \tag{1.1}$$

This system arises when modeling the multistage synthesis of a substance. The dimension n of the system is defined by the number of stages, the first equation describes the intiation law, the last equation does the utilization law, $\theta \geq 0$, τ is the duration of the process, $x_j(t, \tau)$ is the substance concentration at the jth stage, $x_n(t, \tau)$ is the concentration of the final product. Therefore, $x_n(t, \tau)$ is of interest from the practical viewpoint.

It should be noted that systems of the form (1.1) are often termed the 'Goodwin' model [11]. Ordinary differential equations of such kind and more complicated equations arise when modeling gene networks (for example, see [25], the reviews [15, 24] and the bibliography therein).

If n is very large (for instance, the process consists of a great number of the stages) then finding of the last component $x_n(t, \tau)$ of the solution to (1.1) is a very complicated problem. A rigorous mathematical solution to this problem was given by G.V. Demidenko (see [19, Theorems 1–4]). We formulate this result below.

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Suppose that the function $g(t,v) \in C(\overline{\mathbb{R}_2^+})$ is bounded and satisfies the Lipschitz condition with respect to v. Increase the dimension of (1.1) unboundedly and consider the Cauchy problem for each system with the zero initial conditions

$$x_j|_{t=0} = 0, \qquad j = 1, \dots, n. \tag{1.2}$$

Taking only the last component of the solution to each of these Cauchy problems, we obtain the sequence the functions $\{x_n(t,\tau)\}$.

Theorem 1 (G.V. Demidenko). *The sequence $\{x_n(t,\tau)\}$ converges uniformly on every segment $[0,T]$, $T > \tau$:*

$$x_n(t,\tau) \to y(t,\tau), \quad n \to \infty.$$

The limit function $y(t,\tau)$ is a solution to the initial value problem for the delay equation

$$\begin{cases} \dfrac{d}{dt} y(t,\tau) = -\theta y(t,\tau) + g(t-\tau, y(t-\tau,\tau)), \quad t > \tau, \\ y(t,\tau) \equiv 0, \quad 0 \le t \le \tau; \end{cases} \tag{1.3}$$

moreover,

$$\max_{t\in[0,T]} |x_n(t,\tau) - y(t,\tau)| \le \frac{c}{n^{1/4}}, \qquad n > n_0.$$

By Theorem 1, we need not solve the Cauchy problem (1.1), (1.2) for the system of large dimension with large coefficients in order to compute approximately $x_n(t,\tau)$ for $n \gg 1$. It is sufficient to solve only the initial value problem (1.3) for one delay differential equation. This result gives us an effective method for approximate finding $x_n(t,\tau)$ for $n \gg 1$ by using the delay equation; moreover, the estimate established in Theorem 1 characterizes the approximation order.

Theorem 1 has become a basis for deriving similar statements for various systems of nonlinear ordinary differential equations of large dimension (see, for example, [2, 3, 4, 5, 6, 7, 9]). In particular, a perturbation of (1.1) was investigated in [6]. Some examples of the Cauchy problems for (1.1) with nonzero initial conditions were considered in [3]. On the basis of the results, a new method for approximation of solutions to initial value problems for the mentioned delay differential equation with arbitrary initial conditions was proposed in [7]. Three different classes of systems of large dimension were studied in [2, 5], [4] and [9] respectively. In the mentioned works G.V. Demidenko proposed a series of methods for proving limit theorems which establish interconnections between solutions to classes of systems of nonlinear ordinary differential equations of large dimension and generalized solutions to delay differential equations. The readers can be familiarized with some of these methods in the papers [8, 9]. Using the methods, classes of essentially nonlinear systems of large dimension (every equation in the systems is nonlinear) were studied in [16, 22]. It should be noted that there is a number of works devoted to the study of approximation of solutions to delay differential equations by means of solutions to systems of ordinary differential equations of large dimension (see, for instance, [1, 12, 14, 17, 18, 23]). In particular, [17, 23] are the first works in this direction. A brief survey of the literature and the use of the semigroup theory for approximation are given in [1]. Approximation schemes and their development are discussed in [12, 14, 18].

If $\tau \ll 1$ (for example, the synthesis process is very rapid) then the coefficients of (1.1) is very large as well. In [20, 21] we studied the behavior of $x_n(t,\tau)$ in dependence on τ for every fixed n. In particular, the following result was obtained.

Theorem 2. *The sequence* $\{x_n(t,\tau)\}$ *converges uniformly on every segment* $[0,T]$*:*

$$x_n(t,\tau) \to z(t), \qquad \tau \to 0.$$

The limit function $z(t)$ *is a solution to the Cauchy problem*

$$\begin{cases} \dfrac{d}{dt}z = -\theta z + g(t,z), \quad t > 0, \\ z(0) = 0; \end{cases} \tag{1.4}$$

moreover,

$$\max_{t\in[0,T]} |x_n(t,\tau) - z(t)| \le c\,\tau, \qquad \tau \ll 1, \tag{1.5}$$

where $c > 0$ *depends on* θ, G, L, n.

This result gives us an effective method for approximate calculating $x_n(t,\tau)$. Indeed, we may solve the Cauchy problem (1.4) for one ordinary differential equation instead of the Cauchy problem (1.1), (1.2). Then, by the obtained convergence, we have $z(t) \approx x_n(t,\tau)$ for $\tau \ll 1$. Since τ is the duration of the synthesis process, then we can find approximately the concentration $x_n(t,\tau)$ of the final product in the case of very rapid passages from one stage to the other.

More detailed modeling processes of the substance synthesis leads to systems of essentially nonlinear differential equations in comparison with (1.1). As a rule, so-called Hill's type functions are used (for example, see [15]). Our aim is to study one class of systems of such kind described in the next section.

2. Main results

In the present paper we consider the Cauchy problem for the class of systems of nonlinear ordinary differential equations

$$\begin{cases} \dfrac{d\widehat{x}_1}{dt} = g(t,\widehat{x}_n) - \dfrac{n-1}{\tau}\dfrac{\widehat{x}_1}{1+\rho\widehat{x}_1^{\gamma}}, \quad t > 0, \\ \dfrac{d\widehat{x}_j}{dt} = \dfrac{n-1}{\tau}\left(\dfrac{\widehat{x}_{j-1}}{1+\rho\widehat{x}_{j-1}^{\gamma}} - \dfrac{\widehat{x}_j}{1+\rho\widehat{x}_j^{\gamma}}\right), \quad j = 2,\dots,n-1, \\ \dfrac{d\widehat{x}_n}{dt} = \dfrac{n-1}{\tau}\dfrac{\widehat{x}_{n-1}}{1+\rho\widehat{x}_{n-1}^{\gamma}} - \theta\widehat{x}_n, \\ \widehat{x}_j|_{t=0} = 0, \quad j = 1,\dots,n, \end{cases} \tag{2.1}$$

where $\theta \ge 0$, $\tau > 0$, $\rho \ge 0$, $\gamma > 0$. This system arises when modeling the multistage synthesis of a substance as well. Obviously, this system for $\rho = 0$ coincides with (1.1). As was shown in [16], the last component $\widehat{x}_n(t,\tau)$ of the solution to (2.1) for $n \gg 1$ is approximated by the solution to the initial value problem (1.3). Analogous results for a more general class of systems of nonlinear differential equations including the systems of (2.1) were obtained in [22].

We study properties of the components $\widehat{x}_j(t,\tau)$ of the solution to the Cauchy problem (2.1) as functions of t and $\tau \ll 1$, when n is fixed. Assume that the function $g(t,v) \in C(\mathbb{R}_2)$ is nonnegative and bounded $0 \le g(t,v) \le G$ and satisfies the Lipschitz condition

$$|g(t,v_1) - g(t,v_2)| \le L|v_1 - v_2|, \qquad v_1,\ v_2 \in \mathbb{R}.$$

Note that the Cauchy problem (2.1) is uniquely solvable under these conditions; moreover, the components of the solutions are nonnegative (see the detailed proof in [16, 22]).

The main results of the paper are formulated in Theorems 3 and 4 below. Their proofs are given in the next section.

Theorem 3. *The components of the solution to the Cauchy problem* (2.1) *satisfy the estimates*

$$0 \le \widehat{x}_j(t,\tau) < \frac{\tau G(1+\rho)}{n-1}, \quad j = 1,\dots,n-1, \quad t \ge 0, \tag{2.2}$$

for all $\tau < \tau_0$, *where*

$$\tau_0 = \begin{cases} \dfrac{n-1}{G(1+\rho)}, & 0 < \gamma \le 1, \\ \min\left\{\dfrac{n-1}{G(1+\rho)},\ \dfrac{(n-1)}{G(1+\rho)(\rho(\gamma-1))^{1/\gamma}}\right\}, & \gamma > 1. \end{cases}$$

Theorem 4. *The sequence* $\{\widehat{x}_n(t,\tau)\}$ *consisting of the last components of the solutions to the Cauchy problems of the from* (2.1) *converges uniformly on every segment* $[0,T]$*:*

$$\widehat{x}_n(t,\tau) \to z(t), \qquad \tau \to 0. \tag{2.3}$$

The limit function $z(t)$ *is the solution to the Cauchy problem* (1.4).

Corollary 1. *The following estimate holds*

$$\max_{t\in[0,T]} |\widehat{x}_n(t,\tau) - z(t)| \le c_1\tau + c_2\rho(1+\rho)^{1+\gamma}\tau^{1+\gamma}, \qquad \tau \ll 1,$$

where $c_1 > 0$ *depends on* θ, G, L, n, *and* $c_2 > 0$ *depends on* θ, G, L, n, γ.

It follows from Theorem 4 that it is sufficient to solve the Cauchy problem (1.4) for one ordinary differential equation in order to find approximately the last component $\widehat{x}_n(t,\tau)$ of the solution to (2.1) for $\tau \ll 1$. This result gives us an effective method for approximate calculating $\widehat{x}_n(t,\tau)$. Moreover, the less τ, the more exact the method; i.e., the larger the coefficients of the systems of (2.1), the more exact the approximate solution. Thereby this method allows us to avoid difficulties arising inevitably when solving systems of nonlinear differential equations with very large coefficients.

3. The proof of the main results

In this section we prove Theorems 3 and 4.

Proof of Theorem 3. As was mentioned above, it was proved earlier that all components of the solution to the Cauchy problem (2.1) are nonnegative. Therefore it remains to establish the upper estimates of (2.2).

At first we prove this estimate for $j = 1$. Suppose that this is false. Then there are $\tau < \tau_0$ and $t_* > 0$ such that

$$\widehat{x}_1(t,\tau) < \frac{\tau G(1+\rho)}{n-1}, \qquad t \in [0, t_*),$$

$$\widehat{x}_1(t_*,\tau) = \frac{\tau G(1+\rho)}{n-1}.$$

Obviously,

$$\frac{d\widehat{x}_1}{dt}\bigg|_{t=t_*} \geq 0.$$

By (2.1) we have

$$0 \leq \frac{d\widehat{x}_1}{dt}\bigg|_{t=t_*} = g(t_*, \widehat{x}_n) - G\frac{1+\rho}{1+\rho\left(\frac{\tau G(1+\rho)}{n-1}\right)^\gamma} \leq G\left(1 - \frac{1+\rho}{1+\rho\left(\frac{\tau G(1+\rho)}{n-1}\right)^\gamma}\right).$$

The right-hand side of the inequality is negative for $\tau < \dfrac{n-1}{G(1+\rho)}$. Consequently, we arrive at a contradiction.

Prove now the upper estimate of (2.2) for $j \neq 1$. Let $j = 2$. Assume on the contrary that this is false. Then there are $\tau < \tau_0$ and $t_* > 0$ such that

$$\widehat{x}_2(t,\tau) < \frac{\tau G(1+\rho)}{n-1}, \qquad t \in [0, t_*),$$

$$\widehat{x}_2(t_*,\tau) = \frac{\tau G(1+\rho)}{n-1}.$$

Clearly,

$$\frac{d\widehat{x}_2}{dt}\bigg|_{t=t_*} \geq 0.$$

Hence, by (2.1), we obtain

$$\begin{aligned} 0 \leq \frac{d\widehat{x}_2}{dt}\bigg|_{t=t_*} &= \frac{n-1}{\tau}\left(\frac{\widehat{x}_1}{1+\rho\widehat{x}_1^\gamma} - \frac{\widehat{x}_2}{1+\rho\widehat{x}_2^\gamma}\right)\bigg|_{t=t_*} \\ &= \frac{n-1}{\tau}\frac{\widehat{x}_1}{1+\rho\widehat{x}_1^\gamma}\bigg|_{t=t_*} - G\frac{1+\rho}{1+\rho\left(\frac{\tau G(1+\rho)}{n-1}\right)^\gamma}. \end{aligned}$$

Consequently,

$$\frac{\widehat{x}_1}{1+\rho\widehat{x}_1^\gamma}\bigg|_{t=t_*} \geq \frac{\tau G(1+\rho)}{(n-1)\left(1+\rho\left(\frac{\tau G(1+\rho)}{n-1}\right)^\gamma\right)},$$

which is impossible. We now show this.

Consider the case of $0 < \gamma \leq 1$. Since the function

$$f(z) = \frac{z}{1+\rho z^\gamma}, \qquad z \geq 0,$$

is monotone and $\widehat{x}_1(t,\tau)$ satisfies (2.2) for $\tau < \dfrac{n-1}{G(1+\rho)}$, then

$$\frac{\widehat{x}_1}{1+\rho\widehat{x}_1^\gamma}\bigg|_{t=t_*} < \frac{\tau G(1+\rho)}{(n-1)\left(1+\rho\left(\frac{\tau G(1+\rho)}{n-1}\right)^\gamma\right)}.$$

Hence, we arrive at a contradiction. Then (2.2) holds for $\widehat{x}_2(t,\tau)$ for $0<\gamma\le 1$.

Let $\gamma > 1$. In this case $f(z)$ has the maximum at $z_* = \left(\dfrac{1}{\rho(\gamma-1)}\right)^{1/\gamma}$. By (2.2), $0 \le \widehat{x}_1(t_*,\tau) < z_0 = \dfrac{\tau G(1+\rho)}{n-1}$ for $\tau < \dfrac{n-1}{G(1+\rho)}$. Obviously, if $\tau < \dfrac{(n-1)z_*}{G(1+\rho)}$ then $z_0 < z_*$. Consequently, by the definition of $f(z)$, $f(\widehat{x}_1(t_*,\tau)) < f(z_0)$ for $\tau < \min\left\{\dfrac{n-1}{G(1+\rho)}, \dfrac{(n-1)z_*}{G(1+\rho)}\right\}$; i. e.

$$\frac{\widehat{x}_1}{1+\rho\widehat{x}_1^\gamma}\bigg|_{t=t_*} < \frac{\tau G(1+\rho)}{(n-1)\left(1+\rho\left(\frac{\tau G(1+\rho)}{n-1}\right)^\gamma\right)}.$$

Hence, we arrive at a contradiction. Then (2.2) holds for $\widehat{x}_2(t,\tau)$ for $\gamma>1$.

In order to prove (2.2) for $j=3,\dots,n-1$, it suffices to repeat the same arguments as for $\widehat{x}_2(t,\tau)$.

Theorem 3 is proved. □

Proof of Theorem 4. Denote $u(t,\tau)=\widehat{x}(t,\tau)-x(t,\tau)$, where $x(t,\tau)$ is the solution to the Cauchy problem (1.1), (1.2) and $\widehat{x}(t,\tau)$ is the solution to the Cauchy problem (2.1). It is not hard to verify that the vector function $u(t,\tau)$ satisfies the following system of differential equations

$$\frac{du}{dt} = Au + G_1(t) + G_2(t),$$

where A coincides with the matrix of (1.1)

$$A = \begin{pmatrix} -\frac{n-1}{\tau} & 0 & \cdots & \cdots & 0 \\ \frac{n-1}{\tau} & -\frac{n-1}{\tau} & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\frac{n-1}{\tau} & 0 \\ 0 & \cdots & 0 & \frac{n-1}{\tau} & -\theta \end{pmatrix},$$

$$G_1(t) = \begin{pmatrix} g(t,\widehat{x}_n(t,\tau)) - g(t,x_n(t,\tau)) \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$G_2(t) = \rho\frac{n-1}{\tau}\begin{pmatrix} \dfrac{\big(\widehat{x}_1(t,\tau)\big)^{1+\gamma}}{1+\rho\big(\widehat{x}_1(t,\tau)\big)^{\gamma}} \\ \dfrac{\big(\widehat{x}_2(t,\tau)\big)^{1+\gamma}}{1+\rho\big(\widehat{x}_2(t,\tau)\big)^{\gamma}} - \dfrac{\big(\widehat{x}_1(t,\tau)\big)^{1+\gamma}}{1+\rho\big(\widehat{x}_(t,\tau)\big)^{\gamma}} \\ \vdots \\ \dfrac{\big(\widehat{x}_{n-1}(t,\tau)\big)^{1+\gamma}}{1+\rho\big(\widehat{x}_{n-1}(t,\tau)\big)^{\gamma}} - \dfrac{\big(\widehat{x}_{n-2}(t,\tau)\big)^{1+\gamma}}{1+\rho\big(\widehat{x}_{n-2}(t,\tau)\big)^{\gamma}} \\ -\dfrac{\big(\widehat{x}_{n-1}(t,\tau)\big)^{1+\gamma}}{1+\rho\big(\widehat{x}_{n-1}(t,\tau)\big)^{\gamma}} \end{pmatrix}.$$

Taking into account that $u(0,\tau)=0$, we obtain

$$u(t,\tau) = \int_0^t e^{(t-s)A}(G_1(s)+G_2(s))ds.$$

Remind the representation for the matrix exponent [10]

$$e^{tA} = \varphi_1(t)I + \varphi_2(t)(A-\lambda_1 I) + \varphi_3(t)(A-\lambda_1 I)(A-\lambda_2 I) + \dots \\ + \varphi_n(t)(A-\lambda_1 I)\dots(A-\lambda_{n-1}I),$$

where I is the unit matrix, λ_k are the eigenvalues of A,

$$\varphi_1(t) = e^{\lambda_1 t}, \quad \varphi_k(t) = \int_0^t e^{\lambda_k(t-s)}\varphi_{k-1}(s)\,ds, \quad k=2,\dots,n.$$

Obviously, in our case

$$\lambda_1 = -\theta, \quad \lambda_k = -\frac{n-1}{\tau}, \qquad k=2,\dots,n.$$

Consequently,

$$\varphi_1(t) = e^{-\theta t}, \quad \varphi_k(t) = \frac{e^{-\theta t}}{\omega^{k-1}}\left(1 - e^{-\omega t}\sum_{j=0}^{k-2}\frac{(\omega t)^j}{j!}\right), \quad k=2,\dots,n,$$

where $\omega = \dfrac{n-1}{\tau} - \theta$.

Hence, the last component $u_n(t,\tau)$ of $u(t,\tau)$ has the form

$$\begin{aligned} u_n(t,\tau) &= \int_0^t \psi_n(t-s)(g(s,\widehat{x}_n(s,\tau)) - g(s,x_n(s,\tau)))ds \\ &\quad + \int_0^t \sum_{j=1}^n \psi_{n-j+1}(t-s)G_{2j}(s)ds \\ &= J_1(t,\tau) + J_2(t,\tau), \end{aligned} \tag{3.1}$$

where

$$\psi_k(t) = \left(\frac{n-1}{\tau}\right)^{k-1} \varphi_k(t), \qquad k = 1, \dots, n,$$

the functions $G_{2j}(t)$ are the components of the vector function $G_2(t)$.

Consider the first function $J_1(t,\tau)$. Obviously, if $\tau < \tau_1 = \frac{n-1}{\theta}$ then $\psi_k(t)$ satisfy the estimates

$$0 \le \psi_k(t) \le \frac{e^{-\theta t}}{\left(1 - \frac{\theta\tau}{n-1}\right)^{k-1}}, \qquad k = 1, \dots, n, \quad t \ge 0.$$

Consequently, by the Lipschitz condition for $g(t,v)$, we have

$$\begin{aligned} |J_1(t,\tau)| &\le \int_0^t \psi_n(t-s)|g(s,\widehat{x}_n(s,\tau)) - g(s,x_n(s,\tau))|ds \\ &\le \frac{L}{\left(1 - \frac{\theta\tau}{n-1}\right)^{n-1}} \int_0^t |u_n(s,\tau)|ds. \end{aligned} \tag{3.2}$$

Consider the second function $J_2(t,\tau)$. Taking into account the explicit form of $G_{2j}(t)$, we can rewrite $J_2(t,\tau)$ as follows

$$J_2(t,\tau) = \rho \frac{n-1}{\tau} \int_0^t \sum_{j=1}^{n-1} \big(\psi_{n-j+1}(t-s) - \psi_{n-j}(t-s)\big) \frac{\big(\widehat{x}_j(s,\tau)\big)^{1+\gamma}}{1 + \rho\big(\widehat{x}_j(s,\tau)\big)^{\gamma}}\, ds.$$

By Theorem 3,

$$\frac{\big(\widehat{x}_j(t,\tau)\big)^{1+\gamma}}{1 + \rho\big(\widehat{x}_j(t,\tau)\big)^{\gamma}} \le \left(\frac{\tau G(1+\rho)}{n-1}\right)^{1+\gamma}, \quad j = 1, \dots, n, \quad \tau < \tau_0, \quad t \ge 0.$$

To estimate $J_2(t,\tau)$ we use the next lemma.

Lemma 1. *The following estimates hold*

$$\int_0^t |\psi_k(t-s) - \psi_{k-1}(t-s)|\, ds \le \frac{\tau}{n-1} \frac{2}{\left(1 - \frac{\theta\tau}{n-1}\right)^{k-1}},$$

$$k = 2, \dots, n, \qquad \tau < \tau_1 = \frac{n-1}{\theta}, \qquad t \ge 0.$$

Proof. Let $k = 2$. Then,

$$\begin{aligned} |\psi_2(t) - \psi_1(t)| &= \left|\frac{n-1}{\tau}\varphi_2(t) - \varphi_1(t)\right| = \left|\frac{n-1}{\tau}\frac{e^{-\theta t}}{\omega}\left(1 - e^{-\omega t}\right) - e^{-\theta t}\right| \\ &= \left|\left(\frac{1}{1 - \frac{\theta\tau}{n-1}} - 1\right) e^{-\theta t}\left(1 - e^{-\omega t}\right) - e^{-(\theta+\omega)t}\right| \\ &\le \frac{\tau}{n-1} \frac{\theta e^{-\theta t}}{\left(1 - \frac{\theta\tau}{n-1}\right)} + e^{-\frac{n-1}{\tau}t}. \end{aligned}$$

Hence,

$$\begin{aligned}\int\limits_0^t |\psi_2(t-s)-\psi_1(t-s)|\,ds &\le \frac{\tau}{n-1}\frac{1-e^{-\theta t}}{\left(1-\frac{\theta\tau}{n-1}\right)}+\frac{\tau}{n-1}\left(1-e^{-\frac{n-1}{\tau}t}\right)\\ &\le \frac{\tau}{n-1}\frac{2}{\left(1-\frac{\theta\tau}{n-1}\right)}.\end{aligned}$$

Let $k>2$. By definition,

$$\begin{aligned}|\psi_k(t)-\psi_{k-1}(t)| &= \left|\left(\frac{n-1}{\tau}\right)^{k-1}\varphi_k(t)-\left(\frac{n-1}{\tau}\right)^{k-2}\varphi_{k-1}(t)\right|\\ &= \left|\left(\frac{n-1}{\tau}\right)^{k-1}\frac{e^{-\theta t}}{\omega^{k-1}}\left(1-e^{-\omega t}\sum_{j=0}^{k-2}\frac{(\omega t)^j}{j!}\right)\right.\\ &\quad\left.-\left(\frac{n-1}{\tau}\right)^{k-2}\frac{e^{-\theta t}}{\omega^{k-2}}\left(1-e^{-\omega t}\sum_{j=0}^{k-3}\frac{(\omega t)^j}{j!}\right)\right|\\ &= \left|\left(\frac{n-1}{\tau}\right)^{k-1}\frac{e^{-\theta t}}{\omega^{k-1}}\left(1-e^{-\omega t}\sum_{j=0}^{k-2}\frac{(\omega t)^j}{j!}\right)\right.\\ &\quad-\left(\frac{n-1}{\tau}\right)^{k-2}\frac{e^{-\theta t}}{\omega^{k-2}}\left(1-e^{-\omega t}\sum_{j=0}^{k-2}\frac{(\omega t)^j}{j!}\right)\\ &\quad\left.-\left(\frac{n-1}{\tau}\right)^{k-2}\frac{e^{-(\theta+\omega)t}}{\omega^{k-2}}\frac{(\omega t)^{k-2}}{(k-2)!}\right|\\ &= \left|\frac{\tau}{n-1}\frac{\theta e^{-\theta t}}{\left(1-\frac{\theta\tau}{n-1}\right)^{k-1}}\left(1-e^{-\omega t}\sum_{j=0}^{k-2}\frac{(\omega t)^j}{j!}\right)\right.\\ &\quad\left.-\left(\frac{n-1}{\tau}\right)^{k-2}\frac{t^{k-2}}{(k-2)!}e^{-\frac{n-1}{\tau}t}\right|\\ &\le \frac{\tau}{n-1}\frac{\theta e^{-\theta t}}{\left(1-\frac{\theta\tau}{n-1}\right)^{k-1}}+\left(\frac{n-1}{\tau}\right)^{k-2}\frac{t^{k-2}}{(k-2)!}e^{-\frac{n-1}{\tau}t}.\end{aligned}$$

Consequently,

$$\begin{aligned}\int\limits_0^t |\psi_k(t-s)-\psi_{k-1}(t-s)|\,ds \le \frac{\tau}{n-1}&\frac{1-e^{-\theta t}}{\left(1-\frac{\theta\tau}{n-1}\right)^{k-1}}\\ &+\int\limits_0^t\left(\frac{n-1}{\tau}\right)^{k-2}\frac{(t-s)^{k-2}}{(k-2)!}e^{-\frac{n-1}{\tau}(t-s)}\,ds\end{aligned}$$

$$= \frac{\tau}{n-1}\frac{1-e^{-\theta t}}{\left(1-\frac{\theta\tau}{n-1}\right)^{k-1}} + \frac{\tau}{n-1}\left(1-e^{-\frac{n-1}{\tau}t}\sum_{j=0}^{k-2}\frac{\left(\frac{n-1}{\tau}t\right)^j}{j!}\right) \le \frac{\tau}{n-1}\frac{2}{\left(1-\frac{\theta\tau}{n-1}\right)^{k-1}}.$$

The lemma is proved. □

Using this lemma, for $\tau \le \tau_* < \min\{\tau_0, \tau_1\}$, we obtain

$$\begin{aligned} |J_2(t,\tau)| &\le \rho\left(\frac{\tau G(1+\rho)}{n-1}\right)^{1+\gamma}\sum_{j=1}^{n-1}\frac{2}{\left(1-\frac{\theta\tau_*}{n-1}\right)^j} \\ &= 2\rho\left(\frac{\tau G(1+\rho)}{n-1}\right)^{1+\gamma}\left(\frac{1}{\left(1-\frac{\theta\tau_*}{n-1}\right)^{n-1}}-1\right)\frac{n-1}{\theta\tau_*}. \end{aligned} \tag{3.3}$$

By (3.2) and (3.3), for $\tau \le \tau_* < \min\{\tau_0, \tau_1\}$, from (3.1) we have

$$|u_n(t,\tau)| \le M\int_0^t |u_n(s,\tau)|\,ds + K\rho(1+\rho)^{1+\gamma}\tau^{1+\gamma},$$

where

$$M = \frac{L}{\left(1-\frac{\theta\tau_*}{n-1}\right)^{n-1}}, \quad K = 2\left(\frac{G}{n-1}\right)^{1+\gamma}\left(\frac{1}{\left(1-\frac{\theta\tau_*}{n-1}\right)^{n-1}}-1\right)\frac{n-1}{\theta\tau_*}.$$

Applying Gronwall's inequality (for example, see [13]), we obtain

$$|u_n(t,\tau)| \le Ke^{Mt}\rho(1+\rho)^{1+\gamma}\tau^{1+\gamma}.$$

Hence, the following estimate holds

$$|\widehat{x}_n(t,\tau) - x_n(t,\tau)| \le Ke^{MT}\rho(1+\rho)^{1+\gamma}\tau^{1+\gamma} \tag{3.4}$$

on every segment $[0,T]$.

In view of Theorem 2 the sequence $\{x_n(t,\tau)\}$ converges uniformly to the solution $z(t)$ to the Cauchy problem (1.4) on every segment $[0,T]$; moreover, (1.5) holds. Then, from (3.4) we have the uniform convergence

$$\widehat{x}_n(t,\tau) \to z(t), \qquad \tau \to 0, \quad t \in [0,T].$$

Theorem 4 is proved. □

Remark 1. Corollary 1 follows immediately from (1.5) and (3.4).

4. Conclusion

We considered the class of the systems of nonlinear ordinary differential equations with parameters. In particular, systems of such type arise when modeling the multistage synthesis of a substance. We studied properties of the solutions to the systems and proposed a method for approximate solving the systems in the case of very large coefficients. We established the approximation estimates and showed that the convergence rate depends on the parameters characterizing the nonlinearity of the systems. Moreover, the larger the coefficients of the systems, the more exact the approximate solutions. Owing to these causes, this method allows us to avoid difficulties arising inevitably when solving systems of nonlinear differential equations with very large coefficients. As an application, the proposed method can be used for approximate finding the concentration of the final product of the multistage synthesis in the case of very rapid passages from one stage to the other.

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