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Geometry of centers of the polynomial Cauchy–Riemann systems

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Abstracts. Autonomous polynomial systems satisfying the Cauchy–Riemann conditions on the complex plane are studied. Every of them is defined by one complex polynomial. For the fourth degree polynomials whose roots are centers, we prove that they are simple; moreover, either all of them lie on a straight line or three of them form an acute triangle and the fourth root is located at the intersection of its altitudes.

Keywords: autonomous Cauchy–Riemann systems, classification of stationary points, elliptic sector, separatrix, focus, node, center, central configurations

1. Centers of the Cauchy–Riemann systems

We study the dynamics on the Euclidean plane generated by a vector field whose coordinates satisfy the Cauchy–Riemann conditions. The corresponding autonomous system can be written in a compact form:

$$\dot{z} = f(z), \tag{1}$$

where $f(z)$ is an analytical function of a complex variable z. It is clear that such systems have many specific features. For example, if at stationary point z_0 the polynomial resolution of the function $f(z)$ starts with the monomial $A\,(z - z_0)^k$, then the phase portrait of the system around point z_0 is completely determined by the complex coefficient A and natural number k. All possible portraits are presented in our first figure, though there are good reasons to believe that complete classification of all local phase portraits around stationary points of autonomous Cauchy–Riemann systems was well known even by the classics of distant times, when the foundations of the potential theory had been arising, as can be seen, for example, in [3, 4].

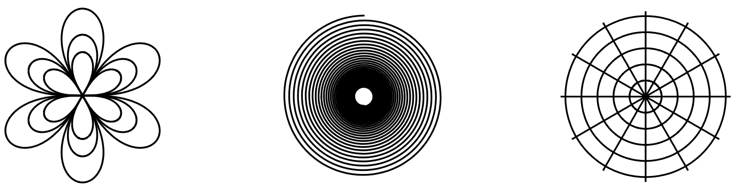

Рис. 1. Stationary points of the Cauchy–Riemann systems

More precisely, if $k \geq 2$ we see a symmetrical rosette of $2(k-1)$ sectors bounded by separatrices, that in turn go out of stationary point and come back to it. When $k = 1$ for simple stationary point there are three cases. If the real part of the derivative $f'(z_0) = A$ is different from zero, we usually get a focus, but if the imaginary part of the coefficient A

is absent, we see a "radiant" node, although experts tend to call it "proper" node. From the point of view of the problem, which this article is devoted to, we are especially interested in the remaining case — the real part of A is zero, the imaginary part is nonzero. We have already seen that only in this case an analytical system can have a center. It is important to emphasize that the reverse is true too: when only this simple condition holds, point z_0 is really a center. So, *an analytical system has a center at a stationary point if and only if the derivative of the field at this point is an imaginary number different from zero.*

However, this is well-known [1, 2, 5] and obvious at the same time. Indeed, assuming for convenience $f(0) = 0$ and $f'(0) = i$, if the reader does not confuse branches of logarithm, it can be noted that

$$\Psi(z) := \int \frac{dz}{f(z)} = \int \frac{dz}{iz + O(z^2)} = \int \frac{1 + O(z)}{iz}\, dz = \frac{\ln z}{i} + O(z) + \text{const}.$$

Let a nonzero solution $z = z(t)$ to our equation at the moment $t = t_0$ be so close to zero that it has survived until the moment $t = t_0 + 2\pi$ and all that time it has never left the neighborhood of zero, where function $\Phi(z)$ is defined. We prove that $z(t_0 + 2\pi) = z(t_0)$.

Since $\dot{\Psi}(z(t)) \equiv 1$, it turns out that $i\Psi(z(t)) = it$ plus a constant, and therefore, when $z = z(t)$ the function of the form $w := ze^{iO(z)}$ is proportional to the exponent e^{it}. It is easy to see that $w = z + O(z^2)$, so z is explicitly expressed by the periodic function w. In particular, $z(t_0 + 2\pi)$ coincides with the initial value $z(t_0)$.

2. Elementary proof

Quite easy to prove the same thing without addressing to multi-valued logarithms. As above, we assume that $f(0) = 0$ and $f'(0) = i$. Therefore,

$$f(z) = iz + O(z^2)$$

as $z \to 0$, where hereafter symbols $O(z^k)$ mean power series that begin no earlier than with degree k. Since the difference

$$\frac{1}{f(z)} - \frac{1}{iz},$$

obviously, does not have singularities at zero, there is one-valued analytical function $\Phi(z)$ around zero, for which

$$\Phi'(z) = \frac{1}{f(z)} - \frac{1}{iz}, \quad \Phi(0) = 0. \tag{2}$$

Greater interest to us is the function

$$w = w(z) = ze^{i\Phi(z)}, \tag{3}$$

Differentiating it and considering (2) we come to conclusion:

$$w'(z) = e^{i\Phi(z)} \left[1 + iz\left(\frac{1}{f(z)} - \frac{1}{iz}\right)\right] = \frac{iw(z)}{f(z)}.$$

Let $z = z(t)$ be a solution to (1) which at some point $t = t_0$ is so close to zero that not only it has survived until the moment $t = t_0 + 2\pi$, but during this period of time it has not left the domain of the function Φ, and so the function w. We need to prove that $z(t_0 + 2\pi)$ coincides with $z(t_0)$. However,

$$\frac{dw(z(t))}{dt} = w'(z(t))\dot{z}(t) = iw(z(t)).$$

Thus, the function $w(z(t))$ is proportional to the exponent e^{it}, and therefore it has the period 2π. It remains to note, considering (2) and (3), that

$$w = w(z) = z(1 + O(z)) = z + O(z^2),$$

hence, by the theorem on the inverse function it follows that around zero

$$z = z(w) = w + O(w^2).$$

It is important that the solution $z(t)$ to (1) is explicitly expressed by the periodic function $w(z(t))$. Thus, $z(t_0 + 2\pi) = z(t_0)$.

3. Dynamically central configurations

Now we can proceed to the question, which this article is actually devoted to. Imagine that on the complex plane we set a finite number of points. We are interested in for what mutual arrangement of these points there exists a polynomial autonomous Cauchy–Riemann system for which only they are its stationary points, and moreover, centers. Such configurations of points we call *dynamically central*, although we can forget about the dynamics, since our question, as we have just seen, has purely algebraic character: *how polynomials with imaginary nonzero derivatives in their roots are constructed*? It is convenient for us to talk about polynomials with the highest coefficient equal to one. However, transition to such polynomials is associated with multiplication by a complex number. This operation maintains the roots of the polynomial, but it can fundamentally change it as a vector field and its appropriate dynamics, because it leads not only to the dilatation that only changes the scale, but also to the rotation that changes the phase portrait beyond recognition. But we have already managed to forget about fields and dynamics. It is important for us that quotients of derivatives of the polynomial in its roots maintain. If all these derivatives are imaginary, then they remain *real proportional*, since the rotation only transfer them from the imaginary axis to another straight line passing through zero. This straight line we call *homogeneous* line, and the polynomial whose derivatives in its roots are different from zero and all lie on the common homogeneous line we call *dynamically homogeneous* polynomial. It is clear that this property of the polynomial do not dependent on its highest coefficient.

Summing up our preparatory conversation, we can notice that *the polynomial is dynamically homogeneous if and only if a set of its the roots is dynamically central.* Thus, our aim is *to describe all dynamically homogeneous polynomials.* In our opinion, it is beautiful and probably difficult problem. Anyway, starting with four points it can hardly be considered as trivial...

4. Configurations of three centers

Linear polynomials are dynamically homogeneous for obvious reason. Each quadratic polynomial with simple roots is also homogeneous, since its derivatives calculated in two its roots differ only in sign, and therefore they are on the common homogeneous line. After multiplying it on an appropriate complex number the corresponding system will have two centers. Starting with the third degree of the polynomial we begin to feel some prohibitions for the configurations we interested in. Let

$$p(z) = (z - z_1)(z - z_2)(z - z_3).$$

The derivatives of the polynomial have the form

$$p'(z_1) = (z_1 - z_2)(z_1 - z_3),$$
$$p'(z_2) = (z_2 - z_1)(z_2 - z_3),$$
$$p'(z_3) = (z_3 - z_1)(z_3 - z_2).$$

We suppose that all the roots are different. Therefore, we can divide:

$$\frac{p'(z_2)}{p'(z_1)} = -\frac{z_2 - z_3}{z_1 - z_3}, \qquad \frac{p'(z_3)}{p'(z_1)} = -\frac{z_3 - z_2}{z_1 - z_2}.$$

If one of the fractions is real, then all three roots lie on the same straight line. Herewith, the other fractions are real too. Conversely, if the roots lie on the same straight line, then all the quotients of derivatives are real. As we can see, *the polynomial of the third degree is dynamically homogeneous if and only if all its three roots lie on the same straight line.* Dynamical interpretation of our observation is the following: cubic Cauchy–Riemann system can have three centers only in the case when its stationary points are simple and lie on the same straight line. For now it is the only form of coexistence of centers.

5. Configurations of four centers

Polynomials of the fourth degree are notably more interesting. Now we will see that there appears another variant of the mutual arrangement of cycles.

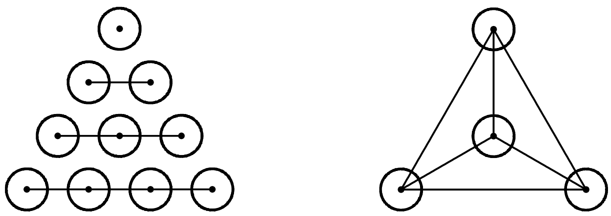

Рис. 2. Dynamically central configurations

Theorem. *If the polynomial of the fourth degree has four different roots and the derivatives in them are real proportional, then there are two variants — either all of them lie on the same straight line or three of them form an acute triangle and a point of the intersection of its altitudes is the place for the fourth root. Conversely, if the roots are arranged as described above, then the derivatives are real proportional.*

We deliberately do not use our terminology in order to simplify the perception of the theorem for the reader. But it can be said in other words: *four complex points are dynamically* *central if and only if all of them lie on the same straight line or they form the configuration of the triangle vertices and the intersection point of its altitudes.* In our second figure in symbolized form there represented all dynamically central configurations which are possible for polynomials of degrees not greater than four.

Proof. If all four roots of the polynomial

$$p(z) = (z - z_1)(z - z_2)(z - z_3)(z - z_4)$$

are different, than its derivatives

$$\begin{aligned} p'(z_1) &= (z_1 - z_2)(z_1 - z_3)(z_1 - z_4), \\ p'(z_2) &= (z_2 - z_1)(z_2 - z_3)(z_2 - z_4), \\ p'(z_3) &= (z_3 - z_1)(z_3 - z_2)(z_3 - z_4), \\ p'(z_4) &= (z_4 - z_1)(z_4 - z_2)(z_4 - z_3) \end{aligned}$$

are nonzero. We can note that it is useful to calculate their pairwise sums. For example,

$$\begin{aligned} p'(z_1) + p'(z_2) &= (z_1 - z_2)^2(z_1 + z_2 - z_3 - z_4), \\ p'(z_1) + p'(z_3) &= (z_1 - z_3)^2(z_1 - z_2 + z_3 - z_4), \\ p'(z_1) + p'(z_4) &= (z_1 - z_4)^2(z_1 - z_2 - z_3 + z_4). \end{aligned}$$

We multiply, say, the second and the third of the last expressions and divide the result by the square of the first of the previous four expressions:

$$\left[1 + \frac{p'(z_3)}{p'(z_1)}\right] \left[1 + \frac{p'(z_4)}{p'(z_1)}\right] = 1 - \left[\frac{z_3 - z_4}{z_1 - z_2}\right]^2.$$

We could come to this formula by using a more "direct" way, but how to guess in advance that we obtain such an expression. If both fractions on the left are real, than the square of the fraction on the right is real too, so the fraction is either real or imaginary. In the first case vectors $z_1 - z_2$ and $z_3 - z_4$ are collinear, in the second case they are orthogonal. It is clear that for all other splits into pairs of four points the conclusions are the same. Hence, there implies what we stated.

Indeed, imagine that on the plane there are four different points. Let their configuration is such that the line passing through any two of them either collinear or orthogonal to the line passing through a pair of other points. If any three points lie on a straight line, then the fourth point lies on it too. If three points form a triangle, then each of three lines passing through the fourth point and one of its vertices must be orthogonal to the opposite side. If our triangle is not right-angled, then its altitudes intersect in the fourth point. If it has an obtuse angle, then we replace its vertex by the fourth point. We obtain an acute triangle and the point that we have excluded now is the intersection point of its altitudes.

However, if one angle is right, then the situation is a little unusual. The fact is that the vertex of this angle is the intersection point of the altitudes of our triangle. Where can the fourth point be? How to agree its presence with the configuration that we are now discussing? Of course, one of the vertices formally could be a twofold stationary point, but it was excluded. But then the hypotenuse must be orthogonal to a segment connecting the fourth point and the vertex of the right angle. Then the segments connecting the fourth point with the vertices of acute angles should be parallel to legs that are opposite to these vertices. We come to an indisputable conclusion — legs are equal and the fourth point together with its three companions represent vertices of a square! To make the "calculation" easier we move its center to the origin, rotate it so that the vertices appear on the real and imaginary axes, and compress or stretch it, if necessary, so that its vertices become ± 1 and $\pm i$. So we get the polynomial $p(z) = z^4 - 1$ whose derivatives $p'(z) = 4z^3$ in its roots are equal to ± 4 or $\mp 4i$ and are not real proportional. In short, the right angle does not occur in the dynamically central configurations of four points.

The proof of the inverse statement may serve as a reward for the reasoning experienced above. If the roots, no matter how many, lie on the same straight line, the quotients of the derivatives can be represented as quotients of equal number of collinear differences of these roots, and therefore, are real. For four points which do not lie on the same straight line we have the "triangular" variant. We calculate the quotients of the derivatives. For example,

$$\frac{p'(z_2)}{p'(z_1)} = \frac{(z_2 - z_1)(z_2 - z_3)(z_2 - z_4)}{(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)} = -\frac{z_2 - z_3}{z_1 - z_4}\,\frac{z_2 - z_4}{z_1 - z_3}.$$

In each fraction on the right there exist all four points, which means that the numerator and denominator as vectors are orthogonal, but as complex numbers are differ by an imaginary multiplier, so the product of these fractions is real. The same thing takes place for other pairs of roots. Theorem is proved. □

It would be exceptionally interesting to know all the configuration of centers for $n \geq 5$. It is not excluded that there exists interesting combinatorics, and full and final answer depends on the arithmetic structure of n. We only emphasize that for any $n \geq 4$ there always exist at least two variants — all the roots of the polynomial p are simple and lie on the same straight line, and "a representative" of another variant may be, for example, the polynomial $p(z) = z(z^{n-1} - 1)$. Just multiplying by a suitable complex number from central Cauchy–Riemann system we easily obtain a system of the same class for which all stationary points are nodes! By the way, in the last example all stationary points are nodes too. To convert them into centers we need only to replace the polynomial $p(z)$ by $ip(z)$. Without any difficulties we could draw full phase portraits for all mentioned examples of autonomous systems on an "extended" complex plane. The pictures are very beautiful and the author hopes that the reader will get a lot of pleasure when he draw them by himself...

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