

УДК 539.3

Numerical method for conformal map building

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Abstracts. Conformal map has application in a lot of areas of science, e.g., fluid flow, heat conduction, solidification, electromagnetic, etc. Especially conformal map applied to elasticity theory can provide most simple and useful solution. But finding of conformal map for custom domain is not trivial problem. We used a numerical method for building a conformal map to solve torsion problem. In addition it was considered an infinite system method to solve the same problem. Results are compared.

Keywords: conformal map, numerical methods, torsion problem, infinite systems

1. Introduction

A conformal (or angle-preserving) map between two domains is a function which preserves oriented angles between curves as well as their direction. Such function preserves both angles and the shapes of infinitesimally small figures, but not necessarily their size. Conformal mapping has for more than a century, been powerful tool in mathematics, engineering, physics and a lot of other subjects of the science, especially in solving various partial differential equations (PDEs).

One major approach in developing methods for numerical conformal mapping is based on the following interpretation of the Riemann mapping theorem: there exists a conformal mapping $f: D \rightarrow U$ with $f(z_0) = 0$ and $f'(z_0)$ nonzero real, where $z_0 \in D$, and this function has a power series expansion $f(z) = c_1(z - z_0) + \sum_{n=2}^{\infty} c_n(z - z_0)^n$, with c_1 nonzero real and $z_0 \in D$, which converges uniformly in every closed disk with center z_0 and contained in D . However, a polynomial which is a good approximation of f in D is not the same as a truncated power series. If a polynomial $p(z) = c'_1(z - z_0) + \sum_{n=2}^N (z - z_0)^n$ approximates f with accuracy $\epsilon > 0$ then it is necessary that every term of $p(z)$ must approximate the corresponding term of the power series with accuracy $\epsilon > 0$ on the set $D \cap B(z_0, R)$, where $R = |z - z_0|$ is the radius of convergence of the power series. All this means is that a polynomial p which is a good approximation of the power series starts in the same way as the power series, but the relative error in the coefficients increases with increasing n [24].

The widely used current computational techniques are based on the integral equation methods where an integral equation is developed to relate the boundaries

of the problem region and the standard region like the unit disk. Once the boundaries are discretized at n points, the integral equation is reduced to a system of algebraic equations. The majority of researches in computational conformal mapping is basically divided in two groups: the first one, where the maps are constructed from a standard region (such as the unit disk) into the problem region, and the second, where the maps are constructed the other way around.

General methods of approximate conformal map building can be found in survey articles by A.F. Bermant and A.I. Markushevich [5], M.K. Govurin and L.V. Kantarovich [13]. Also one can check monographs by L.V. Kantarovich and V.I. Krilov [21], V. Kopenfels and F. Shtalman [23], P.F. Filchakov [11] et al.

There are several types of approximate methods for building the mapping function $z = \omega(\zeta)$ – analytical, graphical-analytical and experimental-analytical. In these methods approximate expression of mapping function is built as polynomial

$$z = \omega(\zeta) = \sum_{k=1}^m C_k \zeta^k \quad (1)$$

where, in general, coefficients $C_k = \alpha + i\beta$ are complex. In general, representation of approximate mapping as a polynomial (1) makes solution of boundary-value problems appreciably easy. The most easy solution of boundary-value problem can be found exactly when the conformal mapping is represented as a polynomial of ζ powers.

There was developed an alternative method for building of interpolation polynomials for simply connected and biconnected regions using Lagrange polynomials. Also it was designed the methodology for constructing of successive approximations with adding intermediate nodes. Description of this method can be found in work by A.G. Ugodchikov [41].

2. Conformal map method

2.1. Problem definition

Let us consider the most simple problem – a problem of approximating to a function $z = \omega(\zeta)$. This function is conformal map from unit disk $|\zeta| < 1$ into the domain S of complex plane $z = x + iy$. The domain S is bounded by piecewise-smooth contour \bar{L} . Let the origin of coordinates for the plane z be inside \bar{L} . We will normalize conformal map in the way such that the center of the unit disk $\zeta = \zeta_0 = 0$ maps to $z = z_0 = 0 \in S$ and the point A_m of the boundary γ of the unit disk with the complex coordinates $\zeta_m = 1$ maps to M_m from the boundary of S (Fig. 1).

Building the conformal map we will keep in mind the theorem [42].

Theorem 1. *Let Σ be finite or infinite simply connected domain on the complex plane $z = x + iy$ with the simple closed boundary. Let $\omega(\sigma)$ be function regular in Σ and continuous up to a boundary. Let point z defined by $z = \omega(\zeta)$ circumscribe simple*

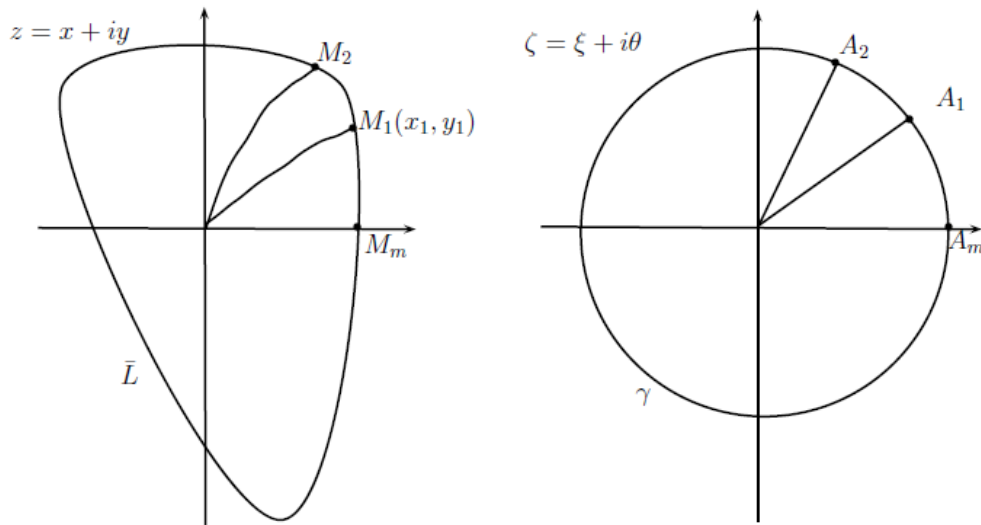


Рис. 1. Conformal map from unit disk $|\zeta| < 1$ into domain S

closed circuit \bar{L} when ζ circumscribe circuit γ . Then relation $z = \omega(\zeta)$ is conformal map from S (enclosed within \bar{L}) into Σ .

A polynomial of positive degrees is a regular function in unit disk $|\zeta| \leq 1$. So we will search for approximation of conformal map that maps unit disk $|\zeta| < 1$ into domain S as a polynomial

$$z = \omega_m(\zeta) = \sum_{k=1}^m C_k \zeta^k. \tag{2}$$

This means that we have to find coefficients $C_k = \alpha_k + i\beta_k (k = 1, \dots, m)$ such that curve L' (with the parametric equation $z = \omega_m(e^{i\theta})$):

- will not have double points and cusps,
- will have set of common points with the boundary \bar{L} ,
- deflection of curve L' from boundary \bar{L} of domain S should be in tolerable limit.

Note that conformity is violated in the corner points. So the exact mapping of corner points (at least two-tangent points) by the polynomial (2) is impossible. Because of this the piecewise smooth boundary \bar{L} should be transformed into curve L with the continuously changing tangent. In general corners of \bar{L} can be rounded by arcs of constant radius. Such rounding can be found in real world, for example in machine elements and structural engineering.

2.2. Construction of conformal map

In [42] it is shown that the problem of conformal map building can be transformed into the problem of construction of interpolation Lagrange polynomial $f_n(\zeta)$. This polynomial can be written as

$$f_n(\zeta) = \sum_{j=1}^m f_j \frac{A(\zeta)}{A'(\zeta_j)(\zeta - \zeta_j)}, \quad (3)$$

where

$$A(\zeta) = \prod_{j=1}^m (\zeta - \zeta_j). \quad (4)$$

If conformal map has next formula

$$\omega_m(\zeta) = \sum_{k=1}^m C_k \zeta^k, \quad (5)$$

where

$$C_k = \alpha_k + i\beta_k = \frac{1}{m} \sum_{j=1}^m z_j e^{-ik\theta_j} \quad (k = 1, \dots, m), \quad (6)$$

then we have formulas for complex coefficients

$$\begin{cases} \alpha_k = \frac{1}{m} \sum_{j=1}^m (x_j \cos \frac{2\pi}{m}kj + y_j \sin \frac{2\pi}{m}kj), \\ \beta_k = \frac{1}{m} \sum_{j=1}^m (y_j \cos \frac{2\pi}{m}kj - x_j \sin \frac{2\pi}{m}kj). \end{cases} \quad (7)$$

After calculation of coefficients C_k ($k = 1, \dots, m$) we should build boundary L' . We should also be sure that curve does not have double points and cusps, and deflection of curve L' from boundary L is in tolerable limit.

We can conclude that in case of known nodes M_j ($j = 1, \dots, m$) from boundary L process of construction of approximate conformal map as an interpolation polynomial is easy.

The problem of match making between nodes of boundaries γ and L is separate and difficult problem by itself. In particular it is nearly impossible to predefine this match exactly. Because of this we will use the approximate methods of determining this match. Another way is algorithm which conjectures the first approximation and then let us obtain more accurate location of such points. To know more please see [42].

3. Method of infinite systems

3.1. Problem definition

Another method of solution of torsion problem is method of infinite systems described in [8].

The problem of torsion of a polygonal-base prism was reduced in [2] to the numerical solution of completely regular infinite systems of linear algebraic equations. Studies on this subject are reviewed in the monograph [3]. The theory of regular and quasiregular infinite systems applied to other problems in the mechanics of elastic bodies is addressed in references [14, 15, 16, 21, 22, 29, 20, 28, 36, 37, 38]. The torsion of a cross-base prism is studied in [1, 3]. Not very accurate solutions of infinite systems allow a satisfactory assessment of the torsional stiffness of a prism, but do not allow a reliable analysis of the stress state, especially in the neighborhood of the vertex of the reentrant angle.

The limitants method was proposed in [22] to estimate solutions of regular infinite systems of linear algebraic equations. The applications of the method are reviewed in references [14, 21, 29, 28]. The use of the limitants method is difficult because of the necessity of solving a great number of finite systems of linear algebraic equations. More attractive is the improved reduction method [14, 15], which leads to one finite system of equations. However, this method does not allow assessing the reliability of approximate solutions. A modification of Koyalovich's limitants method that estimates the upper and lower bounds by solving only two auxiliary systems of linear algebraic equations is proposed in [7]. We will use this method here to solve the problem of torsion of a cross-base prism.

As it is widely known Hooke's law for a prism under torsion can be written in form

$$\theta = \frac{M_T}{C}. \quad (8)$$

The tangential stresses in the prism are expressed in terms of the Prandtl stress function

$$\sigma_{zx} = G\theta \frac{\partial}{\partial y} U; \quad \sigma_{zy} = -G\theta \frac{\partial}{\partial x} U. \quad (9)$$

Which is determined by solving Dirichlet's problem for Poisson's equation

$$\frac{\partial^2}{\partial x^2} U + \frac{\partial^2}{\partial y^2} U = -2; \quad U|_{\Gamma} = 0. \quad (10)$$

in the domain occupied by the prism base.

Following paper [3], we will restrict ourselves to a cross-shaped domain symmetric about the coordinate axes (Fig. 2). The symmetry allows us to consider three subdomains D_0 , D_1 , and D_2 with boundaries dashed (see the Fig. 2).

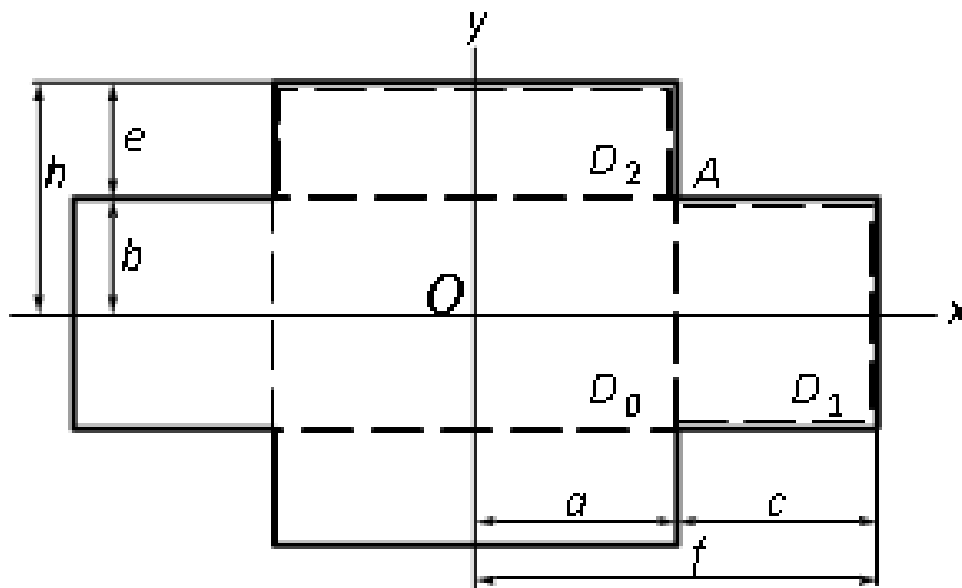


Рис. 2. Subdomains D_0 , D_1 , and D_2

3.2. Representation of the Solution of Dirichlet's Problem

In references [1, 3], a solution was obtained by introducing a system of auxiliary functions. We will outline a different method that leads to somewhat different results.

Using partial solutions of Poisson's equation

$$\begin{aligned} U &= a^2 - x^2, (x, y) \in D_0 \cup D_2, \\ U &= b^2 - y^2, (x, y) \in D_1, \end{aligned} \quad (11)$$

we reduce Dirichlet's problem (10) to Dirichlet's problems for harmonic functions $V_i(x, y)$ in the subdomains D_0, D_1, D_2 :

$$\begin{aligned} \frac{\partial^2}{\partial x^2} V_i + \frac{\partial^2}{\partial y^2} V_i &= 0; (x, y) \in D_i, \\ V_i|_{\Gamma_i} &= F_i(x, y). \end{aligned} \quad (12)$$

The solution of Dirichlet's problem (12) for a harmonic function in a rectangular domain is described in the monograph [21]. In [8] the solution is reduced to a series of special form having the property of Kronecker deltas relative to the values on the sides of the rectangle and formulate this as a lemma. To learn more on this method please read [8].

4. Numerical method

In [42] it is described an algorithm of successive approximation. This algorithm helps to obtain more accurate location of points M_j (see fig. 1) on the boundary L . So this algorithm will let us build the approximate polynomial with the minimum deviation of L' from L .

Described algorithm can be improved. Please note that conformal map which is built by described algorithm can be used for solving of boundary value problems (e.g. in theory of elasticity).

As for theory of elasticity local deflections of boundary L' from boundary L will seriously effect on local stress on boundary. These local disturbances of stress field are caused uppermost not by deflection Δ of boundary L' from given L but they are caused by distortion of radius of curvature. And this is true because boundary L' has form of wave curve which passes main or intermediate interpolation nodes. In case of wave curve the radius of curvature of boundary L' changes in wide range. For complex curve L the radius of curvature can change it's direction twice in the range of single step.

According to [11] the function

$$z = \omega(\zeta) = \sum_{k=1}^m \tilde{C}_k \zeta^k, \quad (13)$$

where

$$\tilde{C}_k = \frac{C_k + C_{k^*}}{2} \quad (k = 1, \dots, m)$$

has same disadvantage. The curve L'' corresponding to function (13) has nearly twice smaller deflection Δ but boundary L'' saves it's form.

Because of local distortions of boundary for coefficients C_k or C_{k^*} as well as for coefficients \tilde{C}_k there are essential errors in determining of stress in boundary points. These errors can reach $50 \div 80\%$ (and even more) as compared with the exact solution for curve L . It is also easy to see that increasing of power of mapping function will not correct this situation.

We can increase the accuracy of boundary L' and accuracy of solution for boundary-value problem by a simple transformation – the integral averaging. We will apply this transformation to approximate solution on interval $\theta - \frac{\pi}{m} \leq \theta \leq \theta + \frac{\pi}{m}$ which is equal to one step of interpolation. So we have:

$$z = \tilde{\omega}_n(\zeta) = \frac{m}{2\pi} \int_{-\frac{\pi}{m}}^{\frac{\pi}{m}} \omega_n[\rho e^{i(\theta+t)}] dt \quad (14)$$

It is good to use here function $\tilde{\omega}_n(\zeta)$ according to (13) because the corresponding curve

L'' has deflections to both sides of the curve L . After integration we get

$$z = \tilde{\omega}_n(\zeta) = \frac{m}{2\pi} \int_{-\frac{\pi}{m}}^{\frac{\pi}{m}} \sum_{k=1}^m \tilde{C}_k \rho^k e^{ik(\theta+t)} dt = \frac{m}{2\pi} \sum_{k=1}^m \left[\tilde{C}_k \zeta^k \int_{-\frac{\pi}{m}}^{\frac{\pi}{m}} e^{ikt} dt \right] = \sum_{k=1}^m C_k \zeta^k \frac{\sin k \frac{\pi}{m}}{k \frac{\pi}{m}} = \sum_{k=1}^m D_k \zeta^k. \quad (15)$$

Here

$$D_k = \tilde{C}_k \sigma_k \quad (k = 1, \dots, m), \quad (16)$$

where σ_k ($k = 1, \dots, m$) – weighting coefficients, which are defined by

$$\sigma_k = \frac{\sin k \frac{\pi}{m}}{\frac{\pi}{m}} \quad (k = 1, \dots, m). \quad (17)$$

The border of L''' , which corresponds to (15), is nearly to match border L . This means that L''' has deflection from L much less than curves L' and L'' . But the biggest advantage is that errors in radius of curvature are not more than $5 \div 10\%$. In the same time curves L' and L'' could not be compared with curve L in the sense of radius of curvature. On the one step of interpolation these curves (L' and L'') could change not only magnitude but also a sign of curvature.

Please note that the operation of integration is applied to all approximate expressions of conforming map. And because of this, in future, we will not distinguish between denotes for coefficients of $\tilde{\omega}_n(\zeta)$ and $\omega_n(\zeta)$. Even after the essential increasing of accuracy of the border of S' (using weighting coefficients (17)) local distortions of the field of stresses partly can be saved.

On the other hand the real machine elements and structural members are made with some tolerance of the form. This means that real boundary does not match the ideal boundary L .

5. Comparison and conclusions

In this article we introduced a numerical method for conformal mapping based on algorithm of successive approximations. In Table 1 a comparison of values of stiffness coefficient calculated by the Chekhov's method described in the chapter 3 (upper C^+ and lower C^- estimates) and values C^K found by conformal maps method (given values are calculated for the cross-shaped domain with rounded corners). The bottom row of the table contains approximate values of C^A from the monograph [3]. In Fig. 3 you can see the relative error estimation graph for conformal map method (solid line) and Abramyan's method (dashed one).

Please note that estimation of stiffness coefficient in [3] is very crude. Maybe this error of estimator ensue from errors in calculation, e.g. accumulated rounding error on

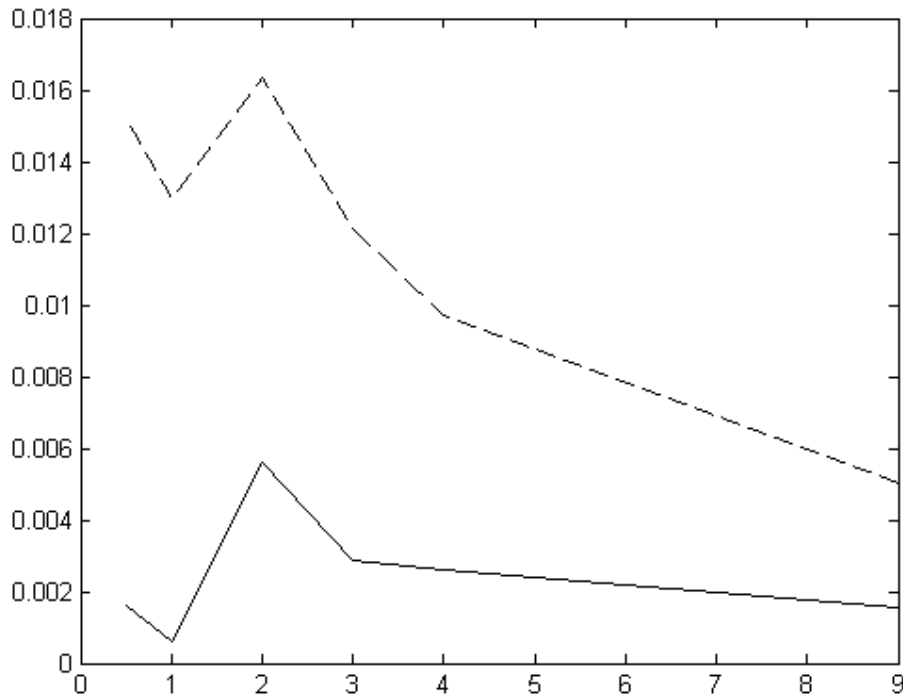


Рис. 3. Relative error estimation

old computers. As for Chekhov’s method we consider its results to compare with results obtained with the help of conformal map method. Chekhov’s estimation is considered as the exact (analytical) solution of the torsion problem for the cross-based (with the right angles) prism.

We should note that estimate C^K obtained by conformal map method is less than lower Chekhov’s estimate C^- which are calculated for the cross-shaped (without

Таблица 1. Comparison of exact and numerical solutions

$\gamma = c/a$	1/2	1	2	3	4	9
$C^+/(16a^4G)$	0.571320	1.064226	1.874217	2.573292	3.246777	6.581903
$C^-/(16a^4G)$	0.571319	1.064225	1.874216	2.573290	3.246775	6.581901
$C^K/(16a^4G)$	0.570416	1.063575	1.863686	2.571791	3.238395	6.571751
$C^A/(16a^4G)$	0.580	1.0504	1.8436	2.5421	3.2152	6.5487

rounded corners) domain. This can be explained by the fact that according to

$$C = 2G \iint_D U dx dy \quad (18)$$

the value of stiffness coefficient is directly proportional to the volume bounded by the surface $z = U(x, y)$.

Also please note that the introduced conformal map method (see Section 2.2) is really convenient. As we said above this method can be easily applied in practical usage, e.g. in engineering calculations. This is because of property of real world machine elements have no right angles. Instead of right angles there is some curve which can be easily approximated by arcs of finite radius.

Another practical advantage of the conformal map method is relatively easy way of program implementation. Convenience in programming is because of modules (libraries with the set of functions) written for the prism with the specific base are easily extended and adapted to new complex domains. Especially the core algorithm functions are not changed for all simply connected domains but it is only changed the algorithm of fetching down of nodes.

And the main advantage of the conformal map method consists of the fact that for most of complex simply connected domains analytical solution cannot be found. Or even if it is found that it is hard to use it in practice. But numerical solution found by conformal map method can be easily found and used (e.g., in engineering calculations)

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Получена 31.05.2010