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Necessary conditions for K-extrema of variational functionals in Sobolev spaces on multi-dimensional domains

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Abstracts. This paper deals with generalized Euler-Ostrogradsky equations and necessary conditions of Legendre-type in the case of compact extrema of variational functionals in Sobolev spaces on multi-dimensional domains. The inverse problem of smoothness refinement for solutions of generalized Euler-Ostrogradsky equations is considered. It is shown that a certain refinement of smoothness of solutions of generalized Euler-Ostrogradsky equations is achieved. Some related problems are considered.

Keywords: variational functional, Sobolev space on domains, compact extremum, generalized Euler-Ostrogradsky equation, generalized Legendre condition.

1. Introduction

Starting with the work by L. Tonelli [22] variational problems in Sobolev spaces attracted much attention by many mathematicians.

In most cases (see, e.g., [7], [9], [11], [12], [13]) the corresponding contributions are connected with so-called direct methods of the calculus of variation which do not use the second variation.

At the same time, the various generalizations of the classical approach were considered. Such way enables to eliminate direct methods (see, e.g., the works by R. Klötzler [14], [15]).

Recently, in our papers [5], [18], [19], [20] a new method of semi-classical type has been developed in the one-dimensional case. It is based on the concept of so-called *compact-analytic* (or, *K-analytic*) properties of variational functionals as well as on the determination of compact extrema of variational functionals in Sobolev spaces.

Subsequently, this method has been extended to the multi-variate case ([16], [17]). Here, following the above mentioned approach fundamental K-extreme necessary conditions for variational functionals acting in Sobolev spaces on multi-dimensional domains are studied.

In Section 2 necessary definitions and theorems of K-analytical properties of variational functionals in Sobolev spaces $W^{1,p}$, $p \in \mathbb{N}$, on a multi-dimensional (Lipschitz) domain D are given. The main results of the paper are contained in Sections 3–5. In Section 3 a generalized Euler-Ostrogradsky equation for compact extrema of variational functionals in Sobolev spaces $W^{1,p}$, $p \in \mathbb{N}$, is established. Next, in Section 4 the inverse problem of smoothness refinement for solutions of generalized Euler-Ostrogradsky equations is considered. It is shown that under some natural conditions a certain refinement of smoothness of K-extremals is achieved.

Finally, in the fifth section a generalized necessary condition of Legendre-type condition for compact extrema of variational functionals in $W^{1,p}$ is derived. At first, the concept of a scalarly non-negative quadratic form is introduced. Then it is proved that the Hessian of the integrand in the leading variable is scalarly non-negative almost everywhere at the point of K–minimum. Moreover, an example is considered as model case.

2. The K-analytical properties of variational functionals in $W^{1,p}(D)$ (review)

Let us introduce now general K-analytic properties for a functional acting on an arbitrary real locally convex space (LCS). In what follows, E is an arbitrary real LCS, $\Phi : E \to \mathbb{R}$ is a real functional, $\mathrm{C}(E)$ is the system of all absolutely convex compacts from E. For any $C \in \mathrm{C}(E)$ denote by E_C the linear hull of C equipped with the Banach norm $\|\cdot\|_C$ generated by the set C. Recall (see [20]) that an arbitrary real Fréchet space E is topologically isomorphic to inductive limit of a spectrum $\{E_C\}_{C\in\mathrm{C}(E)}$ and the expansion

$$E \overset{top}{=} \varinjlim_{C\in\mathrm{C}(E)} E_C \tag{2.1}$$

holds true.

Definition 1. A functional $\Phi : E \to \mathbb{R}$ is called *K-continuously* (*K-differentiable, twice K-differentiable,* etc.) at a point $y \in E$ if all restrictions of Φ to $(y + E_C)$ are continuously (Fréchet differentiable, twice Fréchet differentiable, etc.) at y with respect to the norm $\|\cdot\|_C$. Analogously we say that Φ attains a *compact extremum* (*K-extremum*) at y if all restrictions $\Phi\big|_{y+E_C}$ attain a local extremum at y with respect to the corresponding compact norms in E_C.

Remark 1. 1) All K-properties mentioned above are in general weaker than the usual local ones.

2) Due to expansion (2.1) in case of a Fréchet space E the K-derivatives of any order are multilinear forms of corresponding order which are continuous in the usual sense.

It is well-known (cf. [7], [9]) that well-posedness of the basic variational functional

$$\Phi(y) = \int_D f(x, y, \nabla y)dx \tag{2.2}$$

in Sobolev spaces $W^{1,p}(D)$, $p \in \mathbb{N}$, where D is a compact domain in $\mathbb{R}^n$ with Lipschitz boundary, is usually closely connected with an estimate of the integrand f of type

$$f(x, y, z) \geq \alpha + \beta\|z\|^p, \quad (\beta > 0).$$

Such severe constraints substantially restrict the class of admissible integrands. In our paper [16] we have introduced an essentially larger class of the admissible integrands, so called *K-pseudopolynomials*, using the concept of dominating mixed smoothness (see, for example [21], Chapter 2 and the references given there).

Definition 2. A mapping $f : \mathbb{R}^n_x \times \mathbb{R}_y \times \mathbb{R}^n_z \to \mathbb{R}$ is called *K-pseudopolynomial of the order* $p \in \mathbb{N}$ if it can be represented in the form

$$f(x, y, z) = \sum_{k=0}^{p} R_k(x, y, z)(z)^k\,, \tag{2.3}$$

where the coefficients R_k, taking values in the space of k–linear forms on $\mathbb{R}^n$, are Borel mappings satisfying dominating mixed boundedness in x, y. More precisely, for any compacts $C_x \subset \mathbb{R}^n_x$ and $C_y \subset \mathbb{R}_y$ the coefficients R_k $(k = \overline{0,p})$ are bounded on $C_x \times C_y \times \mathbb{R}^n_z$. For the sake of shortness we shall write $f \in K_p(z)$. Here $(z)^k = (\underbrace{z, \dots, z}_{k})$ is diagonal polyvector in $(\mathbb{R}^n)^k$.

The following theorem shows that the functional (2.2) is well-defined if $f \in K_p(z)$.

Theorem 1. *If the integrand f of the variational functional* (2.2) *belongs to the K-pseudopolynomial class $K_p(z)$ then the functional* (2.2) *is well-defined on the space $W^{1,p}(D)$. Moreover, for any compact set $C_\Delta \subset W^{1,p}(D)$ the estimate $|\Phi(y)| \le \alpha_{C_\Delta} + \beta_{C_\Delta} \cdot (\|y\|_{W^{1,p}})^p$ holds. Here the coefficients $\alpha_{C_\Delta} \ge 0$, $\beta_{C_\Delta} \ge 0$ depend only on the choice of the compact set C_Δ.*

In order to pass to K-continuity conditions for variational functionals in Sobolev spaces a suitable subclass of integrands from $K_p(z)$ will be selected.

Definition 3. Let $f \in K_p(z)$ be continuous. The mapping f is called *a Weierstrass K-pseudopolynomial of p-th order* ($f \in WK_p(z)$) if there exists a representation (2.3) such that all coefficients R_k possess dominating mixed continuity with respect to x, y. More precisely, for any compact sets $C_x \subset \mathbb{R}^n_x$, $C_y \subset \mathbb{R}_y$ the coefficients R_k $(k = \overline{0,p})$ are uniformly continuous and bounded on $C_x \times C_y \times \mathbb{R}^n_z$.

The condition $f \in WK_p(z)$ provides K-continuity of the functional (2.2).

Theorem 2. *If the integrand f of the variational functional* (2.2) *belongs to the Weierstrass class $WK_p(z)$ then the functional $\Phi(y)$ is K-continuous everywhere on the space $W^{1,p}(D)$.*

Furthermore, m-th order K-differentiability is provided by introducing the following K-pseudopolynomial Weierstrass classes.

Definition 4. Let $f \in C^m \cap K_p(z)$. The mapping f is called *a Weierstrass K–pseudopolynomial of the class $W^m K_p(z)$* if there exists a representation (2.3) such that all m-th order jets $(R_k\,, \nabla_{yz} R_k, \dots, \nabla^m_{yz} R_k)$ of the coefficients R_k $(k = \overline{0,p})$ possess dominating mixed smoothness with respect to x, y.

The condition $f \in W^m K_p(z)$ provides m-times K-differentiability of the functional (2.2).

Theorem 3. *If the integrand f of the variational functional* (2.2) *belongs to the class $W^m K_p(z)$, $m \in \mathbb{N}$, then the functional $\Phi(y)$ is m-times K-differentiable on the space $W^{1,p}(D)$. In addition, the classical formula of m-th order variation remains true for the K-variation of m-th order, i. e. it holds*

$$\Phi^{(m)}_K(y)(h)^m = \int\limits_D \left[\sum_{\ell=0}^{m} C^\ell_m \frac{\partial^m f}{\partial y^{m-\ell} \partial z^\ell}(x, y, \nabla y) h^{m-\ell} \cdot (\nabla h)^\ell \right] dx\,. \tag{2.4}$$

Let us emphasize that in cases $m = 1$ and $m = 2$, which are of practical relevance for extremal problems, equality (2.4) reads as

$$\Phi'_K(y)h = \int\limits_D \left[\frac{\partial f}{\partial y}(x, y, \nabla y)h + \frac{\partial f}{\partial z}(x, y, \nabla y) \cdot \nabla h\right] dx\,; \tag{2.5}$$

$$\Phi''_K(y)(h)^2 = \int\limits_D \Big[\frac{\partial^2 f}{\partial y^2}(x, y, \nabla y)h^2 + 2\frac{\partial^2 f}{\partial y \partial z}(x, y, \nabla y)h \cdot \nabla h + \\ + \frac{\partial^2 f}{\partial z^2}(x, y, \nabla y) \cdot (\nabla h)^2\Big] dx\,. \tag{2.6}$$

In the end we want to give a reformulation of Fermat's lemma for K-differentiable functionals (see [6]).

Theorem 4. *Let E be an arbitrary real LCS. Assume that the functional $\Phi : E \to \mathbb{R}$ attains a K-extremum at a point $y \in E$. If Φ is K-differentiable at the point y then $\Phi'_K(y) = 0$.*

3. The generalized Euler-Ostrogradsky equation for K-extremals in $W^{1,p}(D)$

Here we consider the variational functional

$$\Phi(y) = \int_D f(x, y, \nabla y)dx, \quad y(\cdot) \in W^{1,p}(D),\ p \in \mathbb{N}, \tag{3.1}$$

with additional boundary condition

$$y\big|_{\partial D} = y_0\,, \tag{3.2}$$

where $y_0 \in W^{1,p}(\partial D)$, D is a compact domain in $\mathbb{R}^n$ with a Lipschitz boundary ∂D.

Note that the boundary condition (3.2) means, in particular, an additional smoothness of y near ∂D. The definition of Sobolev space $W^{1,p}(\partial D)$ can be found in [8].

To determine the K-extremals of the functional (3.1)–(3.2), we need an almost everywhere-analog of the corresponding classical (C^1) necessary condition, in other words an analog of the Euler-Ostrogradsky equation (see [10]).

Theorem 5. *Let $f \in W^1K_p(z)$. Suppose that*
(i) the functional (3.1) attains a K-extremum at the point $y(\cdot) \in W^{1,p}(D)$,
(ii) the mapping $(\partial f/\partial z)(x, y, \nabla y)$ belongs to the Sobolev space $W^{1,1}(D)$.
Then the generalized Euler-Ostrogradsky equation

$$\frac{\partial f}{\partial y}(x, y, \nabla y) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial z_i}(x, y, \nabla y)\right) = 0 \tag{3.3}$$

holds true almost everywhere (a.e.) on D.
In particular, condition (ii) is fulfilled if

$$\frac{\partial f}{\partial z}(x, y, z) \in C^1(\mathbb{R}^n_x \times \mathbb{R}_y \times \mathbb{R}^n_z) \quad \textit{and} \quad y(\cdot) \in W^{2,p}(D)\,.$$

Proof. 1) If $f \in W^1K_p(z)$ then the functional (3.1) is K-differentiable everywhere in $W^{1,p}(D)$ by virtue of Theorem 3. Therefore, in view of Theorem 4 equality $\Phi'_K(y)h = 0$ holds for any $h \in W^{1,p}(D)$. In detail, this means that

$$\int\limits_D \left[\frac{\partial f}{\partial y}(x,y,\nabla y)h + \frac{\partial f}{\partial z}(x,y,\nabla y)\cdot\nabla h\right]dx = 0 \ \ (\forall\, h \in W^{1,p}(D)). \tag{3.4}$$

2) Now, note that the condition $(\partial f/\partial z)(x,y,\nabla y) \in W^{1,1}(D)$ implies representability of this function by means of indefinite Lebesgue integrals of its partial derivatives with respet to x_i, $i = \overline{1,n}$. Applying now the Green formula [1] to the second summand in (3.4) and taking into account $h|_{\partial D} = 0$ we obtain

$$\begin{aligned}\Phi'_K(y)h &= \int\limits_D \frac{\partial f}{\partial y}(x,y,\nabla y)hdx + \sum_{i=1}^n \left[\int\limits_D \frac{\partial f}{\partial z_i}(x,y,\nabla y)\cdot\frac{\partial h}{\partial x_i}dx\right] = \\ &= \int\limits_D \frac{\partial f}{\partial y}(x,y,\nabla y)hdx + \sum_{i=1}^n \Big[\oint\limits_{\partial D} h\cdot\frac{\partial f}{\partial z_i}(x,y,\nabla y)\cos(\overrightarrow{n},\overrightarrow{e_i})dl - \\ &\qquad - \int\limits_D \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial z_i}(x,y,\nabla y)\right)hdx\Big] = 0\,,\end{aligned} \tag{3.5}$$

where $\overrightarrow{n} = \sum\limits_{k=1}^n \cos(\overrightarrow{n},\overrightarrow{e_k})\overrightarrow{e_k}$ stands for the external normal vector on D. Since the line integral in (3.5) vanishes we get therefrom

$$\Phi'_K(y)h = \int\limits_D \left(\underbrace{\frac{\partial f}{\partial y}(x,y,\nabla y) - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial z_i}(x,y,\nabla y)\right)}_{L(f)(y)}\right)hdx = 0\,. \tag{3.6}$$

3) Next we show that identity (3.6) implies the equality $L(f)(y) = 0$ almost everywhere on D. Assume, on the contrary, $L(f)(y)(x_0) \neq 0$ at some point $x_0 \in D$ of approximate continuity of $L(f)(y)$. Without loss of generality we may suppose $L(f)(y)(x_0) > 0$. Then it follows

$$\int\limits_{\mathcal{O}_\delta(x_0)} L(f)(y)\,dx > 0$$

for $\delta > 0$ small enough, where $\mathcal{O}_\delta(x_0)$ is a δ–neighborhood of the point x_0. We choose $\delta' < \delta$ as much as close to δ such that

$$\int\limits_A L(f)(y)dx < \int\limits_{\mathcal{O}_{\delta'}(x_0)} L(f)(y)dx\,, \tag{3.7}$$

where $A = \mathcal{O}_\delta(x_0)\backslash\mathcal{O}_{\delta'}(x_0)$. Now, we put

$$h(x) = \begin{cases} \quad 1 & ,\text{if} \quad x \in \mathcal{O}_{\delta'}(x_0) \\ \quad 0 & ,\text{if} \quad x \notin \mathcal{O}_\delta(x_0) \\ \text{"radially linear"} & ,\text{if} \quad x \in \mathcal{O}_\delta(x_0)\backslash\mathcal{O}_{\delta'}(x_0) \end{cases} \quad ,h \in W_0^{1,p}(D)\,.$$

We have
$\int\limits_D L(f)(y)hdx =$

$$= \int\limits_{\mathcal{O}_\delta(x_0)} L(f)(y)hdx = \int\limits_A L(f)(y)hdx + \int\limits_{\mathcal{O}_{\delta'}(x_0)} L(f)(y)hdx =: I_1 + I_2. \qquad (3.8)$$

By virtue of (3.7) we get $|I_1| \le \int\limits_A |L(f)(y)|dx < \int\limits_{\mathcal{O}_{\delta'}(x_0)} L(f)(y)dx = I_2$. Together with (3.8) this implies $\int\limits_D L(f)(y)hdx > 0$. The last inequality contradicts condition (3.6).

4) If, in particular, $(\partial f/\partial z)(x,y,z) \in C^1(\mathbb{R}^n_x \times \mathbb{R}_y \times \mathbb{R}^n_z)$ then the mapping f locally satisfies a Lipschitz condition. For $y(\cdot) \in W^{2,p}(D)$ the function $x \mapsto f(x,y,\nabla y)$ belongs to the space $W^{1,p}(D)$. Hence, the composition $(\partial f/\partial z)(x,y,\nabla y)$ belongs to $W^{1,1}(D)$. Thus, condition (ii) of the theorem is fulfilled. □

In what follows solutions of the generalized Euler-Ostrogradsky equation (3.3) with condition (ii) of Theorem 5 are called *K-extremals* of the variational functional (3.1). Note, in addition, that equation (3.3) is satisfied a priori at any point of approximate continuity of ∇y.

4. Smoothness of K-extremals in Sobolev spaces $W^{1,p}$

It is well known that under sufficiently general conditions in the classical C^1-case a solution of the Euler-Ostrogradsky equation even belongs to the class C^2. We look at a similar problem in the Sobolev case (see [4]). Here the question is whether a solution of the generalized Euler-Ostrogradsky equation belongs to the space $W^{2,p}$ under natural conditions. A related problem is whether such a solution possess at least some additional smoothness properties.

Our first result in this direction is not immediately connected with the Euler-Ostrogradsky equation and Sobolev spaces, respectively .

Theorem 6. *Let $f : \mathbb{R}^n_x \times \mathbb{R}_y \times \mathbb{R}^n_z \to \mathbb{R}$, $f \in C^2$, and let the function $y(\cdot) : D \to \mathbb{R}$ be continuous and a.e. differentiable on D. Suppose that*
(i) the gradient $(\partial f/\partial z)(x,y,\nabla y)$ is differentiable a.e. on D;
(ii) the Hessian $(\partial^2 f/\partial z^2)(x,y,\nabla y)$ is non-degenerate a.e. on D, i.e.

$$det\left(\frac{\partial^2 f}{\partial z^2}(x,y,\nabla y)\right) \ne 0 \quad a.\ e.$$

Then the function $y(\cdot)$ is twice approximately differentiable a.e. on D. In addition, it holds

$$\nabla^2_{ap}(y)(x)\cdot\Delta x = \nabla_{ap}(\nabla y)(\Delta x) = \left(\frac{\partial^2 f}{\partial z^2}(x,y,\nabla y)\right)^{-1}\cdot$$
$$\cdot\left[\nabla\left(\frac{\partial f}{\partial z}(x,y,\nabla y)\right)\cdot\Delta x - \frac{\partial^2 f}{\partial z\partial x}(x,y,\nabla y)\cdot\Delta x - (\nabla y,\Delta x)\frac{\partial^2 f}{\partial z\partial y}(x,y,\nabla y)\right]. \qquad (4.1)$$

Proof. 1) We fix $i=\overline{1,n}$ and apply the mean value theorem [23] to the function

$$\frac{\partial f}{\partial z_i}=F_i(x,y,z) \tag{4.2}$$

on the vector interval $[(x,y,z);(x+\Delta x,y+\Delta y,z+\Delta z)]=[h;h+\Delta h]$. It follows

$$\frac{\partial f}{\partial z_i}(x+\Delta x,y+\Delta y,z+\Delta z)-\frac{\partial f}{\partial z_i}(x,y,z)=\left(\nabla\left(\frac{\partial f}{\partial z_i}(\xi)\right),\Delta h\right), \tag{4.3}$$

for some $\xi\in[h;h+\Delta h]$. Because any measurable function is approximately continuous almost everywhere ([3]) we can choose a point $x\in D$ in which ∇y exists and is approximately continuous. Let the conditions (i)–(ii) of the theorem be satisfied. We choose a measurable subset $A_i\subset D$ having x as a density point such that $\nabla y(x+\Delta x)\to\nabla y(x)$ if $x+\Delta x\to x$ in A_i. Now we substitute $\Delta y=y(x+\Delta x)-y(x)$ and $\Delta z==\nabla y(x+\Delta x)-\nabla y(x)$ in (4.3). Moreover, $\Delta y\to 0$ as $\Delta x\to 0$ by continuity of $y(\cdot)$ and $\Delta z\to 0$ as $x+\Delta x\to x$ in A_i in view of approximate continuity of ∇y at the point x. Thus, we obtain

$$\frac{\partial f}{\partial z_i}(x+\Delta x,y+\Delta y,z+\Delta z)-\frac{\partial f}{\partial z_i}(x,y,z)=$$
$$=\left(\frac{\partial^2 f}{\partial z_i\partial x}(\xi),\Delta x\right)+\frac{\partial^2 f}{\partial z_i\partial y}(\xi)\cdot\Delta y+\left(\frac{\partial^2 f}{\partial z_i\partial z}(\xi),\Delta(\nabla y)\right). \tag{4.4}$$

Using notation (4.2) we find

$$F_i(x+\Delta x)-F_i(x)=\left(\frac{\partial^2 f}{\partial z_i\partial x}(\xi),\Delta x\right)+\frac{\partial^2 f}{\partial z_i\partial y}(\xi)\Delta y+\left(\frac{\partial^2 f}{\partial z_i\partial z}(\xi),\Delta(\nabla y)\right). \tag{4.5}$$

Now we determine the principal linear part of (4.5). It holds

$$F_i(x+\Delta x)-F_i(x)=(\nabla F_i,\Delta x)+o(\|\Delta x\|)=\left(\frac{\partial^2 f}{\partial z_i\partial x}(\xi),\Delta x\right)+$$
$$+\frac{\partial^2 f}{\partial z_i\partial y}(\xi)\cdot((\nabla y,\Delta x)+o(\|\Delta x\|))+\left(\frac{\partial^2 f}{\partial z_i\partial z}(\xi),(\nabla_{ap}(\nabla y)\cdot\Delta x+o\left(\|\Delta x\|\right))\right)=$$
$$=\left(\left[\frac{\partial^2 f}{\partial z_i\partial x}(x)+\underbrace{\left(\frac{\partial^2 f}{\partial z_i\partial x}(\xi)-\frac{\partial^2 f}{\partial z_i\partial x}(x)\right)}_{\mathbf{o(1)}}\right],\Delta x\right)+$$
$$+\left[\frac{\partial^2 f}{\partial z_i\partial y}(x)+\underbrace{\left(\frac{\partial^2 f}{\partial z_i\partial y}(\xi)-\frac{\partial^2 f}{\partial z_i\partial y}(x)\right)}_{\mathbf{o(1)}}\right]\cdot((\nabla y,\Delta x)+o(\|\Delta x\|))+$$
$$+\left(\left[\frac{\partial^2 f}{\partial z_i\partial z}(x)+\underbrace{\left(\frac{\partial^2 f}{\partial z_i\partial z}(\xi)-\frac{\partial^2 f}{\partial z_i\partial z}(x)\right)}_{\mathbf{o(1)}}\right],(\nabla_{ap}(\nabla y)\Delta x+o(\|\Delta x\|))\right)=$$

$$= \left(\left[\frac{\partial^2 f}{\partial z_i \partial x}(x) + \alpha(\xi, x)\right], \Delta x\right) + \left[\frac{\partial^2 f}{\partial z_i \partial y}(x) + \beta(\xi, x)\right] \cdot ((\nabla y, \Delta x) + o(\|\Delta x\|))$$
$$+ \left(\left[\frac{\partial^2 f}{\partial z_i \partial z}(x) + \gamma(\xi, x)\right], (\nabla_{ap}(\nabla y)\Delta x + o(\|\Delta x\|))\right),$$

where $\alpha(\xi, x) \to 0$, $\beta(\xi, x) \to 0$, $\gamma(\xi, x) \to 0$ as $\Delta x \to 0$ in $A_i - x$.

Neglecting the small terms we are led to the existence of the approximate gradient of F_i at x and to the equality

$$\underbrace{(\nabla F_i, \Delta x)}_{dF_i(\cdot, \Delta x)} =$$
$$= \left(\frac{\partial^2 f}{\partial z_i \partial x}(x), \Delta x\right) + \frac{\partial^2 f}{\partial z_i \partial y}(x) \cdot (\nabla y, \Delta x) + \left(\frac{\partial^2 f}{\partial z_i \partial z}(x), \nabla_{ap}(\nabla y) \cdot \Delta x\right), \quad (4.6)$$

respectively.

2) Next, we observe that the set $A = \bigcap\limits_{i=1}^{n} A_i$ also has x as its density point. Thus, via the limiting argument $\Delta x \to 0$ in $(A - x)$ we see that all equalities (4.6), $i = \overline{1, n}$, are fulfilled. As a result we have the system

$$\Big\{(\nabla F_i, \Delta x) - \left(\frac{\partial^2 f}{\partial z_i \partial x}(x), \Delta x\right) - \frac{\partial^2 f}{\partial z_i \partial y}(x)(\nabla y, \Delta x) =$$
$$= \left(\frac{\partial^2 f}{\partial z_i \partial z}(x), (\nabla_{ap}(\nabla y), \Delta x)\right)\Big\}_{i=1}^{n}. \quad (4.7)$$

Now, we introduce the matrices

$$A = \left(\nabla F_i - \frac{\partial^2 f}{\partial z_i \partial x}(x) - \frac{\partial^2 f}{\partial z_i \partial y}(x) \cdot \nabla y\right)_{i=1}^{n}, \qquad B = \left(\frac{\partial^2 f}{\partial z_i \partial z}(x)\right)_{i=1}^{n}.$$

Then the system (4.7) can be rewritten as $A \cdot \Delta x = B \cdot (\nabla_{ap}(\nabla y) \cdot \Delta x)$. Therefore it follows $\nabla^2_{ap}(y) \cdot \Delta x = \nabla_{ap}(\nabla y) \cdot \Delta x = B^{-1} \cdot (A \cdot \Delta x) = (B^{-1} \cdot A) \cdot \Delta x$, that is

$$\nabla^2_{ap}(y)(x) \cdot \Delta x = \nabla_{ap}(\nabla y)(x) \cdot \Delta x = \left(\frac{\partial^2 f}{\partial z^2}(x, y, \nabla y)\right)^{-1} \cdot$$
$$\cdot \left[\nabla\left(\frac{\partial f}{\partial z}(x, y, \nabla y)\right) \cdot \Delta x - \frac{\partial^2 f}{\partial z \partial x}(x, y, \nabla y) \cdot \Delta x - (\nabla y, \Delta x)\frac{\partial^2 f}{\partial z \partial y}(x, y, \nabla y)\right]. \quad (4.8)$$

The last expression can be rewritten in matrix form as

$$\nabla^2_{ap}(y)(x) = \left(\frac{\partial^2 f}{\partial z^2}(x, y, \nabla y)\right)^{-1} \cdot$$
$$\cdot \left[\nabla\left(\frac{\partial f}{\partial z}(x, y, \nabla y)\right) - \frac{\partial^2 f}{\partial z \partial x}(x, y, \nabla y) - (\nabla y, \cdot)\frac{\partial^2 f}{\partial z \partial y}(x, y, \nabla y)\right]. \quad (4.9)$$

□

As an application we can establish a result on some strengthening of smoothness of K-extremals in Sobolev spaces.

Corollary 1. *Under the assumptions of Theorem 6 let the function* $y(\cdot) \in W^{1,p}(D)$, $y\big|_{\partial D} = y_0$, *be a* K*-extremal of the functional* (3.1)*. Then, at all points* $x \in D$ *of both approximate continuity of the gradient* $\nabla y(x)$ *and non-degeneracy of the Hessian* $(\partial^2 f/\partial z^2)(x, y, \nabla y)$ *the trace function*

$$Tr\left(\frac{\partial^2 f}{\partial z^2}(x, y, \nabla y) \cdot \nabla^2_{ap}(y)(x)\right)$$

is approximately continuous as well. Moreover, equality

$$Tr\left(\frac{\partial^2 f}{\partial z^2}(x, y, \nabla y) \cdot \nabla^2_{ap}(y)(x)\right) = \\ = \frac{\partial f}{\partial y}(x, y, \nabla y) - Tr\left(\frac{\partial^2 f}{\partial z \partial x}(x, y, \nabla y) + (\nabla y, \cdot) \cdot \frac{\partial^2 f}{\partial z \partial y}(x, y, \nabla y)\right) \tag{4.10}$$

holds true.

Proof. Multiplying both parts of equality (4.9) by $(\partial^2 f/\partial z^2)(x, y, \nabla y)$ from the left side we get

$$\frac{\partial^2 f}{\partial z^2}(x, y, \nabla y) \cdot \nabla^2_{ap}(y)(x) = \\ = \nabla\left(\frac{\partial f}{\partial z}(x, y, \nabla y)\right) - \frac{\partial^2 f}{\partial z \partial x}(x, y, \nabla y) - (\nabla y, \cdot) \cdot \frac{\partial^2 f}{\partial z \partial y}(x, y, \nabla y)\,. \tag{4.11}$$

Applying the trace operator to both sides of (4.11), we obtain the equality

$$Tr\left(\nabla\left(\frac{\partial f}{\partial z}(x, y, \nabla y)\right)\right) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial z_i}(x, y, \nabla y)\right).$$

This yields equality (4.10) using the generalized Euler-Ostrogradsky equation (3.3). In view of the approximate continuity of right-hand side of (4.10) the trace function

$$Tr\left((\partial^2 f/\partial z^2)(x, y, \nabla y) \cdot \nabla^2_{ap}(y)(x)\right)$$

is approximately continuous at those points where the gradient $\nabla y(x)$ is approximately continuous as well. □

The previous results can be essentially improved under the assumption of usual almost everywhere continuity of the gradient of K-extremal. In particular, we can show the usual repeated almost everywhere differentiability of K-extremal (i.e., almost everywhere differentiability of usual gradient of K-extremal).

Theorem 7. *Under the assumptions of Theorem 6 let the gradient $\nabla y(x)$ be continuous almost everywhere on D. Then the following statements hold true.*
(i) There exists $\nabla^2(y)(x)$ a.e. on D. Moreover, we have

$$\nabla^2(y)(x) = \left(\frac{\partial^2 f}{\partial z^2}(x, y, \nabla y)\right)^{-1} \cdot \\ \cdot \left(\nabla \left(\frac{\partial f}{\partial z}(x, y, \nabla y)\right) - \frac{\partial^2 f}{\partial z \partial x}(x, y, \nabla y) - (\nabla y, \cdot) \frac{\partial^2 f}{\partial z \partial y}(x, y, \nabla y)\right). \quad (4.12)$$

(ii) It holds the formula

$$Tr\left(\frac{\partial^2 f}{\partial z^2}(x, y, \nabla y) \cdot \nabla^2(y)(x)\right) = Tr\left(\nabla\left(\frac{\partial f}{\partial z}(x, y, \nabla y)\right)\right) - \\ - Tr\left(\frac{\partial^2 f}{\partial z \partial x}(x, y, \nabla y) + (\nabla y, \cdot) \cdot \frac{\partial^2 f}{\partial z \partial y}(x, y, \nabla y))\right) \quad (4.13)$$

for the trace function $Tr\left((\partial^2 f/\partial z^2)(x, y, \nabla y) \cdot \nabla^2(y)(x)\right)$.
(iii) If, in particular, $y(\cdot)$ satisfies the generalized Euler-Ostrogradsky equation (3.3)*, then the function $Tr\left((\partial^2 f/\partial z^2)(x, y, \nabla y) \cdot \nabla^2(y)(x)\right)$ is also continuous at all points of continuity of $\nabla y(x)$ and the trace formula* (4.10) *can be rewritten as*

$$Tr\left(\frac{\partial^2 f}{\partial z^2}(x, y, \nabla y) \cdot \nabla^2(y)(x)\right) = \\ = \frac{\partial f}{\partial z}(x, y, \nabla y) - Tr\left(\frac{\partial^2 f}{\partial z \partial x}(x, y, \nabla y) + (\nabla y, \cdot) \cdot \frac{\partial^2 f}{\partial z \partial y}(x, y, \nabla y))\right). \quad (4.14)$$

Proof. (i) Extracting the principal linear part in (4.5) and passing to the limit as $\Delta x \to 0$ arbitrarily we find the system of equations

$$\Big\{(\nabla F_i, \Delta x) - (\frac{\partial^2 f}{\partial z_i \partial x}(x), \Delta x) - \frac{\partial^2 f}{\partial z_i \partial y}(x) \cdot (\nabla y, \Delta x) = \\ = \left(\frac{\partial^2 f}{\partial z_i \partial x}(x), \nabla(\nabla y) \cdot \Delta x\right)\Big\}_{i=1}^{n}. \quad (4.15)$$

Equality (4.12) follows immediately from (4.15).

(ii) Multiplying both sides of (4.12) by $(\partial^2 f/\partial z^2)(x, y, \nabla y) \cdot \nabla^2(y)(x)$ from the right side and passing to traces we easily obtain equality (4.13).

(iii) Suppose, in particular, that $y(\cdot)$ satisfies the generalized Euler-Ostrogradsky equation (3.3). Then equality (4.13) can be rewritten as (4.14). Moreover, the function

$$Tr\left((\partial^2 f/\partial z^2)(x, y, \nabla y) \cdot \nabla^2(y)(x)\right)$$

is continuous simultaneously with $\nabla y(x)$. $\square$

Now, we consider an important special case of the integrand which leads to explicit representations of weighted and usual Laplacians of K-extremals $y(\cdot)$.

Corollary 2. *Under the assumptions of Theorem 7 let the integrand f be given as*

$$f(x,y,z) = P(x,y) + Q(x,y)\cdot(z) + R(x,y)\cdot(z)^2 .$$

Suppose that both the coefficients

$$P : D\times\mathbb{R}_y \to \mathbb{R},\ Q : D\times\mathbb{R}_y \to L_1(\mathbb{R}^n_z) \cong \mathbb{R}^n,\ R : D\times\mathbb{R}_y \to L_2(\mathbb{R}^n_z) \cong M_n(\mathbb{R}),$$

(where $M_n(\mathbb{R})$ is set of $n\times n$ matrices on $\mathbb{R}$) and the gradient ∇y are continuous almost everywhere in D. Then the following statements hold true.
(i) The trace function (4.13) *can be represented as*

$$Tr\left(R(x,y)\cdot\nabla^2(y)(x)\right) = Tr\left(\nabla(\frac{\partial f}{\partial z}(x,y,\nabla y))\right) - \\ - Tr\left(\frac{\partial^2 f}{\partial z\partial x}(x,y,\nabla y) + (\nabla y,\cdot)\cdot\frac{\partial^2 f}{\partial z\partial y}(x,y,\nabla y))\right). \quad (4.16)$$

In addition, in the special case of a diagonal matrix $R(x,y) = diag\left(\rho_{ii}(x,y)\right)_{i=1}^n$ representation (4.16) *can be rewritten as*

$$Tr\left(R(x,y)\cdot\nabla^2(y)(x)\right) = \sum_{i=1}^n \rho_{ii}(x,y)\cdot\frac{\partial^2 y}{\partial x_i^2} =: \Delta_\rho y(x) = \\ = Tr\left(\nabla(\frac{\partial f}{\partial z}(x,y,\nabla y))\right) - Tr\left(\frac{\partial^2 f}{\partial z\partial x}(x,y,\nabla y) + (\nabla y,\cdot)\cdot\frac{\partial^2 f}{\partial z\partial y}(x,y,\nabla y))\right). \quad (4.17)$$

Here $\Delta_\rho y$ denotes the weighted Laplacian of y with the weights $\{\rho_{ii}(x,y)\}_{i=1}^n$. In particular, in case of the unit matrix $R(x,y)\equiv E$ we obtain the representation of the usual Laplacian Δy

$$\Delta y(x) = Tr\left(\nabla(\frac{\partial f}{\partial z}(x,y,\nabla y))\right) - \\ - Tr\left(\frac{\partial^2 f}{\partial z\partial x}(x,y,\nabla y) + (\nabla y,\cdot)\cdot\frac{\partial^2 f}{\partial z\partial y}(x,y,\nabla y))\right). \quad (4.18)$$

(ii) Let $y(\cdot)$ be a K-extremal of the functional (3.1) *and let the gradient ∇y be continuous a.e. on D. Then both the weighted Laplacian $\Delta_\rho y$ and the usual Laplacian Δy are continuous a.e. on D as well. Moreover, the right-hand sides of representations* (4.17)–(4.18) *read as*

$$\frac{\partial f}{\partial y}(x,y,\nabla y) - Tr\left(\frac{\partial^2 f}{\partial z\partial x}(x,y,\nabla y) + (\nabla y,\cdot)\cdot\frac{\partial^2 f}{\partial z\partial y}(x,y,\nabla y))\right). \quad (4.19)$$

Proof. The corollary is an immediate consequence of Theorem 7. □

We have shown that the solutions of the generalized Euler-Ostrogradsky equation possess some additional smoothness properties. However, in spite of this fact the question whether or not K-extremals belong to the Sobolev space $W^{2,p}$ must be answered negatively in general. Let us discuss an appropriate counter example.

Example 1. Let us consider the most simple variational functional

$$\Phi(y) = \int\limits_D |\nabla y(x)|^2 dx, \quad \big(y(\cdot) \in W^{1,2}(D),\ D = [0;1] \times [0;1]\big).$$

Here $f(x,y,z) = (y_{x_1})^2 + (y_{x_2})^2$, $\dfrac{\partial f}{\partial y} = 0$; $\dfrac{\partial f}{\partial z_1} = 2y_{x_1}$; $\dfrac{\partial f}{\partial z_2} = 2y_{x_2}$; $\dfrac{\partial}{\partial x_1}\left(\dfrac{\partial f}{\partial z_1}\right) = 2y_{x_1x_1}$; $\dfrac{\partial}{\partial x_2}\left(\dfrac{\partial f}{\partial z_2}\right) = 2y_{x_2x_2}$. Hence, the generalized Euler-Ostrogradsky equation reads as

$$y_{x_1x_1} + y_{x_2x_2} \overset{a.e.}{=} 0. \tag{4.20}$$

Let $\chi(t)$ be the "Cantor ladder"on $[0;1]$ (see, e.g., [2]). We put

$$y_0(x) = \int\limits_0^{x_1} \chi(t)dt + \int\limits_0^{x_2} \chi(t)dt, \qquad 0 \le x_1 \le 1, \quad 0 \le x_2 \le 1.$$

Then $(y_0)_{x_1} = \chi(x_1)$; $(y_0)_{x_2} = \chi(x_2)$; $(y_0)_{x_1x_1} = \chi'(x_1) = 0$ a.e. on $[0;1] \subset \mathbb{R}_{x_1}$; $(y_0)_{x_2x_2} = \chi'(x_2) = 0$ a.e. on $[0;1] \subset \mathbb{R}_{x_2}$. Hence, $y_0(\cdot)$ satisfies the generalized Euler-Ostrogradsky equation (4.20). Nevertheless, $y_0(\cdot) \notin W^{2,2}(D)$ because $\nabla y_0(\cdot) \notin W^{1,2}(D)$. Thus, in contrast to the classical C^1-case an essential strengthening of smoothness of K-extremals in Sobolev case does not occur.

5. Generalized Legendre necessary conditions for K-extrema

Definition 5. Let φ be a quadratic form acting on a real vector space E. We call φ the *scalarly non-negative form* ($\varphi \overset{scal}{\ge} 0$) if the condition $\varphi < 0$ is not fulfilled, that is, if there exists $h \in E$ ($h \ne 0$) such that $\varphi(h) \ge 0$.

Theorem 8. *Assume that the variational functional* (3.1) *attains a K-minimum at $y(\cdot) \in W^{1,p}(D)$. Moreover, suppose that*
(i) the integrand f belongs to the Weierstrass class $W^2K_p(z)$;
(ii) the mapping $\left(\partial^2 f/\partial y \partial z\right)(x,y,\nabla y)$ belongs to the Sobolev space $W^{1,1}(D)$.
Then the generalized Legendre necessary condition

$$\frac{\partial^2 f}{\partial z^2}(x, y(x), \nabla y(x)) \overset{scal}{\ge} 0 \tag{5.1}$$

is fulfilled for the K-extremal $y(\cdot)$ almost everywhere on D.

Proof. We transform the canonical expression of the second K-variation of Φ

$$\Phi''_K(y)(h)^2 = \int\limits_D \Big[\frac{\partial^2 f}{\partial y^2}(x,y,\nabla y)h^2 + 2\sum_{i=1}^n \frac{\partial^2 f}{\partial y \partial z_i}(x,y,\nabla y) \cdot h \cdot \frac{\partial h}{\partial x_i} + \\ + \frac{\partial^2 f}{\partial z^2}(x,y,\nabla y) \cdot (\nabla h)^2\Big] dx \qquad \big(h \in W^{1,p}(D)\big). \tag{5.2}$$

To this end we use the repeated K-differentiability of Φ ([17], Theorem 2) and condition (ii) of the theorem. Application of Green's formula ([1]) to the second summand on the right-hand side in (5.2) and taking into account the boundary condition

$$(y|_{\partial D} = y_0) \Rightarrow (h|_{\partial D} = 0)$$

we obtain

$$\begin{aligned}\Phi''_K(y)(h)^2 &= \int\limits_D \frac{\partial^2 f}{\partial y^2}(x,y,\nabla y)h^2 + \sum_{i=1}^n \Bigg[\oint\limits_{\partial D} h^2 \cdot \frac{\partial^2 f}{\partial y_i \partial z_i}(x,y,\nabla y)\cos(\overrightarrow{n}, \overrightarrow{e_i})dl - \\ &\quad - \int\limits_D \frac{\partial}{\partial x_i}\left(\frac{\partial^2 f}{\partial y \partial z}(x,y,\nabla y)\right) h^2 dx \Bigg] + \int\limits_D \frac{\partial^2 f}{\partial z^2}(x,y,\nabla y)\cdot(\nabla h)^2 dx = \\ &= \int\limits_D \Bigg(\left[\frac{\partial^2 f}{\partial y^2}(x,y,\nabla y) - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial^2 f}{\partial y \partial z}(x,y,\nabla y)\right)\right] h^2 + \\ &\quad + \int\limits_D \frac{\partial^2 f}{\partial z^2}(x,y,\nabla y)\cdot(\nabla h)^2 \Bigg) dx\,. \qquad (5.3)\end{aligned}$$

Here $\overrightarrow{n} = \sum\limits_{k=1}^n \cos(\overrightarrow{n}, \overrightarrow{e_k})e_k$ stands for the external normal vector on ∂D. We give the proof by contradiction. Suppose that (5.1) is not fulfilled. Then there exists a subset $\widetilde{A^1_0} \subset D$, $\mu\widetilde{A^1_0} > 0$ such that inequality

$$\varphi(x) := \frac{\partial^2 f}{\partial z^2}(x, y(x), \nabla y(x)) < 0$$

holds for any $x \in \widetilde{A^1_0}$. The last inequality can be replaced by

$$\psi(x,\widetilde{h}) = \varphi(x)\cdot(\widetilde{h})^2 < 0$$

for any $x \in \widetilde{A^1_0}$, $\widetilde{h} \in \mathbb{R}^n_z$, $\|\widetilde{h}\| = 1$. Next, we choose a compact subset $A^1_0 \subset \widetilde{A^1_0}$ with positive measure $\mu A^1_0 > 0$. Applying the Weierstrass theorem to the function $\psi(x,\widetilde{h})$ on the compact set $A^1_0 \times (\|\widetilde{h}\| = 1)$ we obtain the inequality

$$\psi(x,\widetilde{h}) \le -k_0 < 0 \quad (\forall\, x \in A^1_0\,,\ \|\widetilde{h}\| = 1)\,.$$

Hence, using the second order homogeneity of ψ in $\widetilde{h}$ it follows immediately

$$\psi(x,\widetilde{h}) \le -k_0\cdot\|\widetilde{h}\|^2 \quad (\forall\, x \in A^1_0\,,\ \widetilde{h} \in \mathbb{R}^n_z)\,.$$

Here k_0 does not depend on the choice of $x \in A^1_0$ and $\widetilde{h} \in \mathbb{R}^n_z$. In particular, this implies

$$\frac{\partial^2 f}{\partial z^2}(x, y(x), \nabla y(x))\cdot(\nabla h(x), \nabla h(x)) \le -k_0^2\cdot\|\nabla h(x)\|^2 \quad (x \in A^1_0)\,.$$

Now, we choose a set $A^2_0 \subset D$ with $\mu A^2_0 > \mu D - \mu A^1_0$ such that the inequality

$$\frac{\partial^2 f}{\partial y^2}(x,y,\nabla y) - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial^2 f}{\partial y \partial z}(x,y,\nabla y)\right) \le C_0^2 < \infty$$

holds for all $x \in A_0^2$. Then the set $A_0 := A_0^1 \cap A_0^2$ has a positive measure as well.

Next, let x_0 be an arbitrary density point of A_0 ([3]). We choose a neighborhood $\mathcal{O}_{\delta_0}(x_0)$ ($\delta_0 > 0$) such that the inequality

$$\frac{\mu(A_0 \cap \mathcal{O}_\delta(x_0))}{\mu(\mathcal{O}_\delta(x_0))} > 1 - \varepsilon_0 \qquad (0 < \varepsilon_0 < 1)$$

holds for $\delta < \delta_0$. Now we define a function $h_0(x)$ by

$$h_0(x) = \begin{cases} \sqrt{\delta} & , \text{as } x = x_0; \\ 0 & , \text{as } \|x - x_0\| \geq \delta; \\ \text{is "radially linear"} & , \text{as } \|x - x_0\| < \delta. \end{cases}$$

Then, in a δ-neighborhood $0 < \|x - x_0\| < \delta$ we get

$$h_0^2 \leq \delta, \; \nabla h_0 = (\pm\frac{1}{\sqrt{\delta}}, \ldots, \pm\frac{1}{\sqrt{\delta}}). \tag{5.4}$$

Combining (5.4) and (5.3) we find

$$\Phi''_K(y)(h_0)^2 = \int\limits_D \Big(\left[\frac{\partial^2 f}{\partial y^2}(x, y, \nabla y) - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\frac{\partial^2 f}{\partial y \partial z}(x, y, \nabla y) \right) \right] h_0^2 +$$

$$+ \frac{\partial^2 f}{\partial z^2}(x, y, \nabla y)(\nabla h_0, \nabla h_0) \Big) dx \leq \int\limits_{\mathcal{O}_\delta(x_0)} \Big(\Big[\frac{\partial^2 f}{\partial y^2}(x, y, \nabla y) -$$

$$- \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\frac{\partial^2 f}{\partial y \partial z}(x, y, \nabla y) \right) \Big] h_0^2 + \frac{\partial^2 f}{\partial z^2}(x, y, \nabla y)(\nabla h_0, \nabla h_0) \Big) dx \leq$$

$$\leq C_0^2 \cdot \delta \cdot [(1 - \varepsilon_0) \cdot 2\delta + [(1 - \varepsilon_0) \cdot 2\delta \cdot \frac{h}{\delta} \cdot (-k_0^2) =$$

$$= 2C_0^2 (1 - \varepsilon_0) \cdot \delta^2 + 2 \cdot (1 - \varepsilon_0) \cdot h \cdot (-k_0^2) < 0$$

for $\delta > 0$ small enough. Finally, using Taylor's formula of second order in direction h_0 we immediately obtain the inequality

$$\Phi(y + th_0) - \Phi(y) < 0$$

for $t > 0$ small enough. Hence, Φ does not realize a minimum on any absolutely convex compact $C_\varepsilon \subset W^{1,p}(D)$ for which $C_\varepsilon \bigcap \mathbb{R} \cdot h_0 \neq \{0\}$ holds. Therefore, Φ does not possess a K-minimum at the point $y(\cdot)$. The last result contradicts the assumption of the theorem. □

At the end, let us consider an example of a two-dimensional variational functional having a non-smooth K-extremal but satisfying the generalized Legendre necessary condition.

Example 2. We put

$$\Phi(y) = \int_{-1}^1 \int_{-1}^1 \left(\int_0^{\sqrt{y_{x_1}^2 + y_{x_2}^2}} cos^2 t^2 dt \right) dx \; (y \in W^{1,2}(D), D = [-1; 1] \times [-1; 1]).$$

1. In this case we have

$$f(z_1,z_2)=\int_0^{\sqrt{z_1^2+z_2^2}} cos^2 t^2 dt.$$

Thus we obtain

$$1)\qquad \frac{\partial f}{\partial z_i}=\frac{z_i}{\sqrt{z_1^2+z_2^2}}\cdot\cos^2(2(z_1^2+z_2^2));$$

$$2)\qquad \frac{\partial^2 f}{\partial z_i^2}=\cos^2(z_1^2+z_2^2)\frac{z_i^2}{\sqrt{(z_1^2+z_2^2)^3}}-\sin 2(z_1^2+z_2^2)\frac{2z_i^2}{\sqrt{z_1^2+z_2^2}};$$

$$\frac{\partial^2 f}{\partial z_i\partial z_j}=-\cos^2(z_1^2+z_2^2)\frac{z_i\cdot z_j}{\sqrt{(z_1^2+z_2^2)^3}}-\sin 2(z_1^2+z_2^2)\frac{2z_i\cdot z_j}{\sqrt{z_1^2+z_2^2}}$$

$$(i,j=1,2;\ i\neq j).$$

We introduce the mapping

$$\varphi(z_i,z_j)=\frac{f(z_1,z_2)}{z_1^2+z_2^2}\ .$$

Since

$$\varphi(\infty)=\varphi'_{z_i}(\infty)=\varphi''_{z_iz_j}(\infty)=0\quad(i,j=1,2)$$

the jet $(\varphi,\partial\varphi/\partial z,\partial^2\varphi/\partial z^2)$ possesses dominating mixed continuity. Hence $f\in W^2K_2(z)$. Moreover, $(\partial f/\partial z)(x,y,\nabla y)\in W^{1,1}(D)$.

2. It is evident that $\Phi(y)$ attains a minimum at any point $y(\cdot)\in W^{1,2}$ satisfying the condition

$$|\nabla y|^2=y_{x_1}^2+y_{x_2}^2=\frac{\pi}{2}+\pi k\quad\text{almost everywhere}\quad(k\in\mathbb{Z}).$$

We consider the special point of minimum $y_0(x_1,x_2)=\sqrt{\frac{\pi}{4}}(|x_1|+|x_2|)$. In this case the generalized Euler-Ostrogradsky equation reads as

$$\frac{\partial}{\partial x_1}\left(\frac{z_1}{\sqrt{z_1^2+z_2^2}}\cdot\cos^2(2(z_1^2+z_2^2))\right)+\frac{\partial}{\partial x_2}\left(\frac{z_2}{\sqrt{z_1^2+z_2^2}}\cdot\cos^2(2(z_1^2+z_2^2))\right)\overset{a.e.}{=}0\,. \tag{5.5}$$

Because of

$$\frac{\partial y_0}{\partial x_i}=\sqrt{\frac{\pi}{4}}sgn\,x_i\ \ (i=1,2),\qquad |\nabla y_0|^2=\left(\frac{\partial y_0}{\partial x_1}\right)^2+\left(\frac{\partial y_0}{\partial x_2}\right)^2=\frac{\pi}{2}\quad\text{a.e.}$$

the function $y_0(\cdot)$ satisfies equation (5.5).

3. Finally, in the case under consideration we obtain

$$\frac{\partial^2 f}{\partial z^2}(x,y_0(x),\nabla y_0(x))\overset{a.e.}{=}$$

$$=\begin{pmatrix}\frac{y_{x_2}^2\cos^2|\nabla y|^2}{|\nabla y|^3}-\frac{2y_{x_1}^2\sin 2|\nabla y|^2}{|\nabla y|} & -\frac{y_{x_2}y_{x_1}\cos^2|\nabla y|^2}{|\nabla y|^3}-\frac{2y_{x_1}y_{x_2}\sin 2|\nabla y|^2}{|\nabla y|}\\ -\frac{y_{x_2}y_{x_1}\cos^2|\nabla y|^2}{|\nabla y|^3}-\frac{2y_{x_1}y_{x_2}\sin 2|\nabla y|^2}{|\nabla y|} & \frac{y_{x_1}^2\cos^2|\nabla y|^2}{|\nabla y|^3}-\frac{2y_{x_2}^2\sin 2|\nabla y|^2}{|\nabla y|}\end{pmatrix}.$$

This implies

$$\frac{\partial^2 f}{\partial z^2}(x,y_0(x),\nabla y_0(x))\overset{a.e.}{=}2\pi\cdot\begin{pmatrix}0&0\\0&0\end{pmatrix}\overset{scal}{\geq}0\ \text{ a.e. on } D$$

for the K-extremal $y_0(\cdot)$. Thus, y_0 satisfies the generalized Legendre necessary condition but it does not satisfy the usual one because of nonsmoothness.

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