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Necessary conditions of topological conjugacy for three-dimensional diffeomorphisms with heteroclinic tangencies[1]

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Abstracts. In present paper we consider a class of 3-dimensional diffeomorphisms with finite hyperbolic chain recurrent set and finite number of orbits of heteroclinic tangencies. We prove that necessary conditions for topological conjugacy of two diffeomorphisms from this class is a generalization of moduli of stability for analogous two-dimensional systems.

Keywords: topological conjugacy, heteroclinic tangencies, moduli of stability.

Introduction

According to S. Newhouse and J. Palis [11], there is an open set of arcs that start in Morse-Smale diffeomorphism and have first bifurcation point at diffeomorphism with heteroclinic tangency. In survey [1] bifurcations of systems from boundary of set of Morse-Smale type diffeomorphisms are described; this boundary includes systems with non-transversal intersections of invariant manifolds. Obviously, heteroclinic tangency of invariant manifolds is not structurally stable situation. Moreover, in such situation continuous topological invariants (*moduli of stability*) appear.

J. Palis was one of the first who noticed existence of moduli of stability [13]. He discovered that even two-dimensional diffeomorphisms with heteroclinic one-sided tangency already have moduli. Further advance in this direction was done by W. de Melo and S. J. van Strien in [8] where they found necessary and sufficient conditions for diffeomorphism of orientable surface to have finite moduli of topological stability; these moduli fully describe all classes of topological conjugacy in some neighbourhood of such diffeomorphisms.

T.M. Mitryakova and O.V. Pochinka obtained a topological classification for a class of orientable surface diffeomorphisms with finite numbers of moduli of stability [9]. Radical difference between result of this paper and paper [8] is that the classification was done not only for "near" systems from some neighbourhood, but for "far" systems too.

There are only few results known in case of higher dimensions. In S.Newhouse, J.Palice and F.Takens' paper [12] has been proven a necessary condition for topological conjugacy of two diffeomorphisms with one orbit of one-sided heteroclinic tangency. In J.Palis and W. de Melo's paper [6] n-dimensional manifolds' diffeomorphisms with one orbit of one-sided heteroclinic tangency are considered and classification of diffeomorhphisms in neighbourhood is presented.

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In present paper we study necessary conditions for topological conjugacy of 3-manifolds' diffeomorphisms with few orbits of one-sided heteroclinic tangency.

1. Formulation of results

In present paper we consider a class of diffeomorphisms $\Psi \subset \mathrm{Diff}^4(M^3)$. We say that orientation preserving diffeomorphism of smooth manifold M^3 is from class Ψ if it satisfies following conditions:

1) chain reccurrent set $\mathcal{R}_f$ is finite and consists of hyperbolic fixed points. Eigenvalues of $\mathrm{D}f$ at fixed points are positive and have no resonances[2] until third order;

2) wandering set of diffeomorphism f contains finite number of heteroclinic tangency orbits.

Let p, q be different hyperbolic saddle points of diffeomorphism f such that intersection $W^s_p \cap W^u_q$ is non-empty. Any point $x \in W^s_p \cap W^u_q$ is called *a point of heteroclinic intersection.* Further characterizing of point x is based on whether intersection is *transversal* or *non-transversal.* Two smooth submanifolds N_1 and N_2 ($N_1,\ N_2 \subseteq M^3$) *are intersected transversally at point* $x \in (N_1 \cap N_2)$ if $T_xN_1 + T_xN_2 = T_xM^3$. Let x be an isolated tangency point of two-dimensional manifolds N_1 and N_2, $N_1, N_2 \subset M^3$; then x is a *one-sided tangency point* if there exists neighbourhood V_x of point x such that N_2 intersects not more than one connected component of $V_x \setminus N_1$. For example, any isolated point of tangency of two-dimensional invariant manifolds of 3-dimensional diffeomorphism f is one-sided tangency point.

Let σ be a saddle fixed point of diffeomorphism $f \in \Psi$. Denote by $J_\sigma : \mathbb{R}^3 \to \mathbb{R}^3$ a linear diffeomorphism defined by Jordan normal form of linearization $\mathrm{D}f$ in neighbourhood of σ. The origin $O(0,0,0)$ is a saddle point of J_σ. In section 2 we construct examples of J_σ-invariant neighbourhood of point O for each type of Jordan form.

Definition 1. We say that f-invariant neighbourhood U_σ of saddle fixed point σ is C^1-linearizing if there exists C^1-diffeomorphism $\psi_\sigma : U_\sigma \to U_{J_\sigma}$ that conjugates $f\Big|_{U_\sigma}$ with $J_\sigma\Big|_{U_{J_\sigma}}$.

The following lemma is proven in section 2

Lemma 1. *For any saddle fixed point* σ *of diffeomorphism* $f \in \Psi$ *exists linearizing neighbourhood.*

We say that point a is in $\mathcal{A}$ if it is a point of heteroclinic tangency of two-dimensional invariant manifolds. For any point $a \in \mathcal{A}$ we define saddle points σ^s_a and σ^u_a such that $a \in W^s_{\sigma^s_a} \cap W^u_{\sigma^u_a}$. Obviously saddle point σ^s_a has one-dimensional unstable manifold and σ^u_a has one-dimensional stable manifold. Denote by μ_a and λ_a eigenvalue that corresponds to one-dimensional eigenspace for $J_{\sigma^s_a}$ and for $J_{\sigma^u_a}$ respectively.

[2]Recall that the set of eigenvalues $(\rho_1, \rho_2, \ldots, \rho_n)$ of operator A is called *resonant* if exist non-negative integers $i \in \{1, \ldots, n\}$, $m_j\ (j = 1, \ldots, n)$, $|m| = \sum\limits_{j=1}^{n} m_j \geq 2$ such that $\rho_i = \rho_1^{m_1} \cdot \rho_2^{m_2} \cdot \ldots \cdot \rho_n^{m_n}$. The number $|m|$ is called *an order of the resonance.*

For any point a we define parameter Θ_a and put it equal to $\dfrac{\ln \mu_a}{\ln \lambda_a}$. The following theorem has been proven in article [12] in general setting for manifolds of dimension greater or equal than 2. For sake of completeness we prove it in our case.

Theorem 1. *Suppose that $f, f' \in \Psi$ are topologically conjugated via homeomorphism h such that $h(a) = a'$ for point $a \in \mathcal{A}$, $h(\sigma_a^s) = \sigma_{a'}^s$, $h(\sigma_a^u) = \sigma_{a'}^u$. Then $\Theta_a = \Theta_{a'}$.*

Recall that $U_{\sigma_a^s} = \psi_{\sigma_a^s}^{-1}(U_{J_{\sigma_a^s}})$ and $U_{\sigma_a^u} = \psi_{\sigma_a^u}^{-1}(U_{J_{\sigma_a^u}})$ are linearizing neighbourhoods. Denote by U_a the connected component of $U_{\sigma_a^s} \cap U_{\sigma_a^u}$ that contains point a. For any point $p \in U_a$ put by definition

$$p^s = \psi_{\sigma_a^s}(p) = ([p]_x^s, [p]_y^s, [p]_z^s),$$
$$p^u = \psi_{\sigma_a^u}(p) = ([p]_x^u, [p]_y^u, [p]_z^u),$$
$$g_a = \psi_{\sigma_a^u} \circ \left(\psi_{\sigma_a^s}\Big|_{U_a} \right)^{-1} : \psi_{\sigma_a^s}(U_a) \to \psi_{\sigma_a^u}(U_a).$$

Coordinate expression of map g_a is

$$g_a(x, y, z) = (\xi_a(x, y, z), \eta_a(x, y, z), \chi_a(x, y, z)).$$

Now consider a subclass $\Psi^* \subset \Psi$ with diffeomorphisms such that Θ_a is irrational for any point $a \in \mathcal{A}$. Let a and d be points from $\mathcal{A}$ such that $\sigma_d^s = \sigma_a^s$, $\sigma_d^u = \sigma_a^u$, and derivatives $\beta_a = \dfrac{\partial \chi_a}{\partial z}(a^s)$ and $\beta_d = \dfrac{\partial \chi_d}{\partial z}(d^s)$ have same sign. By definition, put $\tau_d^a = |\beta_a / \beta_d|^{1/\ln \mu_a}$. The main result of present paper is the following theorem:

Theorem 2. *Suppose that $f, f' \in \Psi^*$ are topologically conjugated via homeomorphism h such that $h(a) = a'$, $h(d) = d'$ for points $a, d \in \mathcal{A}$ such that $\beta_a \cdot = \beta_d > 0$, $h(\sigma_a^s) = \sigma_{a'}^s$ and $h(\sigma_a^u) = \sigma_{a'}^u$. Then $\tau_d^a = \tau_{d'}^{a'}$.*

2. Linearizing neighborhood

Recall that by $J_\sigma : \mathbb{R}^3 \to \mathbb{R}^3$ we denote linear diffeomorphism defined by Jordan normal form of linearization $\mathrm{D}f$ at saddle point σ. Suppose σ has two-dimensional stable manifold; then there are three possibilities for diffeomorphism $J_\sigma : \mathbb{R}^3 \to \mathbb{R}^3$ and J_σ-invariant neighbourhood U_{J_σ} of origin:

1. $J_\sigma(x, y, z) = (\lambda_1 x, \lambda_2 y, \mu z)$, where $0 < \lambda_1, \lambda_2 < 1$ and $\mu > 1$;
   $U_{J_\sigma} = \left\{ (x, y, z) \in \mathbb{R}^3 : \left(x|z|^{-\log_\mu \lambda_1} \right)^2 + \left(y|z|^{-\log_\mu \lambda_2} \right)^2 \le 1 \right\}.$
2. $J_\sigma(x, y, z) = (\lambda x + y, \lambda y, \mu z)$, where $0 < \lambda < 1$ and $\mu > 1$;
   $U_{J_\sigma} = \left\{ (x, y, z) \in \mathbb{R}^3 : |z|^{-2\log_\mu \lambda} \cdot \left(y^2 - \dfrac{y}{\lambda \ln \mu} \cdot \ln |z| + x \right)^2 \le 1 \right\} \bigcup \{z = 0\}.$
3. $J_\sigma(x, y, z) = \big(\rho \cdot (x \cdot \cos\varphi - y \cdot \sin\varphi), \rho \cdot (x \cdot \sin\varphi + y \cdot \cos\varphi), \mu z\big)$, where $0 < \rho < 1$ and $\mu > 1$;
   $U_{J_\sigma} = \left\{ (x, y, z) \in \mathbb{R}^3 : (x^2 + y^2) \cdot |z|^{-\log_\mu \rho} \le 1 \right\}.$

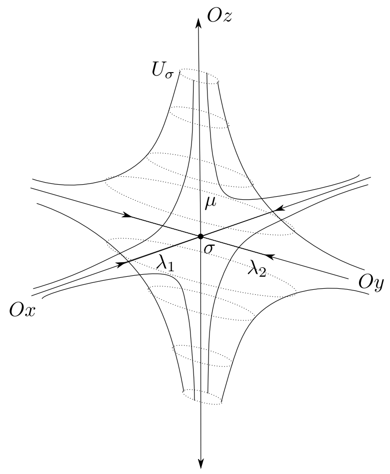

Рис. 1. Linearizing neighborhood U_{J_σ} for $J_\sigma(x, y, z) = (\lambda_1 x, \lambda_2 y, \mu z)$

Similar formulas can be written in case when saddle fixed point σ has two-dimensional unstable manifolds.

Lemma 2. *Any saddle fixed point σ of diffeomorphism $f \in \Psi$ has linearizing neigbourhood.*

Proof. According to Belitskii' theorem (see [2], chapter 6, §5 or [15], theorem 3.20), since $f \in \mathrm{Diff}^4(M^3)$ and has no resonances until third order, in neighbourhood of σ diffeomorphism f is conjugated with its differential via C^1 coordinate transformation. In other words, for map $f \in \Psi$ exist neighbourhood V_σ of saddle point σ, neighbourhood V_O of origin $O(0, 0, 0)$, and C^1-diffeomorphism $\bar{\psi}_\sigma : V_\sigma \to V_O$ which conjugates $f|_{V_\sigma}$ with $\mathrm{D}f_\sigma|_{V_O}$. Put by definition $\tilde{V}_\sigma = \bigcup\limits_{n\in\mathbb{Z}} f^n(V_\sigma)$ and $\tilde{V}_O = \bigcup\limits_{n\in\mathbb{Z}} \mathrm{D}f^n_\sigma(V_O)$. Since W^s_σ and W^u_σ are submanifolds of M^3 (this can be done like in [10], theorem 1), diffeomorphism $\bar{\psi}_\sigma$ extends to a diffeomorphism $\tilde{\psi}_\sigma : \tilde{V}_\sigma \to \tilde{V}_O$; we put by definition $\tilde{\psi}_\sigma(x) = \mathrm{D}f^{-m}_\sigma(\bar{\psi}_\sigma(f^m(x)))$ where m is an integer number such that $f^m(x) \in V_\sigma$. Diffeomorphism $\mathrm{D}f_\sigma|_{\tilde{V}_O}$ is conjugated with its Jordan form J_σ by linear coordinate transformation $S : \mathbb{R}^3 \to \mathbb{R}^3$.

For any $k \in \mathbb{N}$ put by definition

$$U^k_{J_\sigma} = \Big\{(x, y, z) \in \mathbb{R}^3 : (\sqrt{k}x, \sqrt{k}y, z) \in U_{J_\sigma}\Big\}.$$

Choose $k \in \mathbb{N}$ such that $U^k_{J_\sigma} \subset \tilde{V}_O$. Note that $J_\sigma\Big|_{U^k_{J_\sigma}}$ is conjugated with $J_\sigma\Big|_{U_{J_\sigma}}$ by diffeomorphism $h(x, y, z) = (\sqrt{k}x, \sqrt{k}y, z)$. It now follows that $U_\sigma = \tilde{\psi}^{-1}_\sigma \circ S^{-1}(U^k_{J_\sigma})$ is a required linearizing neighbourhood with conjugacy

$$\psi_\sigma = h \circ S \circ \tilde{\psi}_\sigma : U_\sigma \to U_{J_\sigma}.$$

□

3. Auxiliary statements

Lemma 3. *For the map* $g_a(x,y,z) = (\xi_a(x,y,z), \eta_a(x,y,z), \chi_a(x,y,z))$ *the following relations holds:* $\dfrac{\partial \chi_a}{\partial x}(a^s) = 0$, $\dfrac{\partial \chi_a}{\partial y}(a^s) = 0$, $\dfrac{\partial \chi_a}{\partial z}(a^s) \neq 0$.

Proof. There is a following correspondence between $W^s_{\sigma^s_a}$, $W^u_{\sigma^u_a}$ and their images in linearizing neighbourhoods $U_{J_{\sigma^s_a}}$, $U_{J_{\sigma^u_a}}$:

- plane $Oxy \in U_{J_{\sigma^s_a}}$ corresponds to $W^s_{\sigma^s_a}$
- surface $\psi_{\sigma^s_a}(W^u_{\sigma^u_a})$ in $U_{J_{\sigma^s_a}}$ corresponds to $W^u_{\sigma^u_a}$;
- plane $Oxy \in U_{J_{\sigma^u_a}}$ corresponds to $W^u_{\sigma^u_a}$;
- surface $\psi_{\sigma^u_a}(W^s_{\sigma^s_a})$ in $U_{J_{\sigma^u_a}}$ corresponds to $W^s_{\sigma^s_a}$.

Let a be the tangency point of $W^s_{\sigma^s_a}$ and $W^u_{\sigma^u_a}$. Since $\psi_{\sigma^s_a}$ and $\psi_{\sigma^u_a}$ are diffeomorphisms, points $\psi_{\sigma^s_a}(a)$ and $\psi_{\sigma^u_a}(a)$ will be heteroclinic tangency points of images of $W^s_{\sigma^s_a}$ and $W^u_{\sigma^u_a}$ in neighbourhoods $U_{J_{\sigma^s_a}}$ and $U_{J_{\sigma^u_a}}$ (see [7]).

Now consider two smooth curves on plane $Oxy \subset U_{J_{\sigma^s_a}}$ that pass through point a^s. Let the tangent vectors to these curves at point a^s be equal to $(1,0,0)$ and $(0,1,0)$ respectively. Map $g_a(x,y,z)$ sends these curves to curves on surface $\psi_{\sigma^u_a}(W^s_{\sigma^s_a}) \subset U_{J_{\sigma^u_a}}$. Tangent vectors to curve images at point a^u must have zero z-coordinate because curves touch plane $Oxy \subset U_{J_{\sigma^u_a}}$ at point a^u. Suppose that curves were parameterized by a parameter t; then by chain rule we obtain

$$\begin{pmatrix} \xi'_t \\ \eta'_t \\ \chi'_t \end{pmatrix} = \begin{pmatrix} \dfrac{\partial \xi_a}{\partial x} & \dfrac{\partial \xi_a}{\partial y} & \dfrac{\partial \xi_a}{\partial z} \\ \dfrac{\partial \eta_a}{\partial x} & \dfrac{\partial \eta_a}{\partial y} & \dfrac{\partial \eta_a}{\partial z} \\ \dfrac{\partial \chi_a}{\partial x} & \dfrac{\partial \chi_a}{\partial y} & \dfrac{\partial \chi_a}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} x'_t \\ y'_t \\ z'_t \end{pmatrix},$$

where the matrix of partial derivatives is a Jacobi matrix for map $g_a(x,y,z)$. Substituting tangent vector in right hand side for $(1,0,0)$ and $(0,1,0)$, we obtain that tangent vectors to curve images are equal to $(\dfrac{\partial \xi_a}{\partial x}, \dfrac{\partial \eta_a}{\partial x}, \dfrac{\partial \chi_a}{\partial x})$ and $(\dfrac{\partial \xi_a}{\partial y}, \dfrac{\partial \eta_a}{\partial y}, \dfrac{\partial \chi_a}{\partial y})$ respectively. Since curve images touch plane $Oxy \subset U_{J_{\sigma^u_a}}$ we get that $\dfrac{\partial \chi_a}{\partial x} = 0$ and $\dfrac{\partial \chi_a}{\partial y} = 0$. However, $g_a(x,y,z)$ is a diffeomorphism and $\det \mathrm{D}g_a(a^s) \neq 0$, so we necessarily have that $\dfrac{\partial \chi_a}{\partial z} \neq 0$. □

In further theorems and lemmas we will often refer to the following sentence:

Proposition 1. Let σ be a saddle fixed point and J_σ be one of the Jordan forms mentioned earlier. Then for any sequence $\{r_n\}$, $r_n \in U_{J_\sigma} \setminus Oz$ that tends to $r \in (Oz \setminus O)$ exist subsequence $\{r_{n_j}\}$, sequence $\{k_j\}$, $k_j \to +\infty$ and point $q \in Oxy \setminus \{O\}$ such that $\{f^{k_j}(r_{n_j})\}$ tends to point q (the proof is analogous to the proof of lemma 2.1.1 in [3]).

Let $\{a_\nu\} \subset (U_a \setminus W^u_{\sigma^u_a})$ be the sequence of points such that $\{a_\nu\}$ tends to $a \in W^s_{\sigma^s_a} \cap W^u_{\sigma^u_a}$ as $n \to +\infty$ and there exist positive constants C_1 and C_2 such that $\left|\dfrac{[a_\nu]^s_x - [a]^s_x}{[a_\nu]^s_z}\right| < C_1$

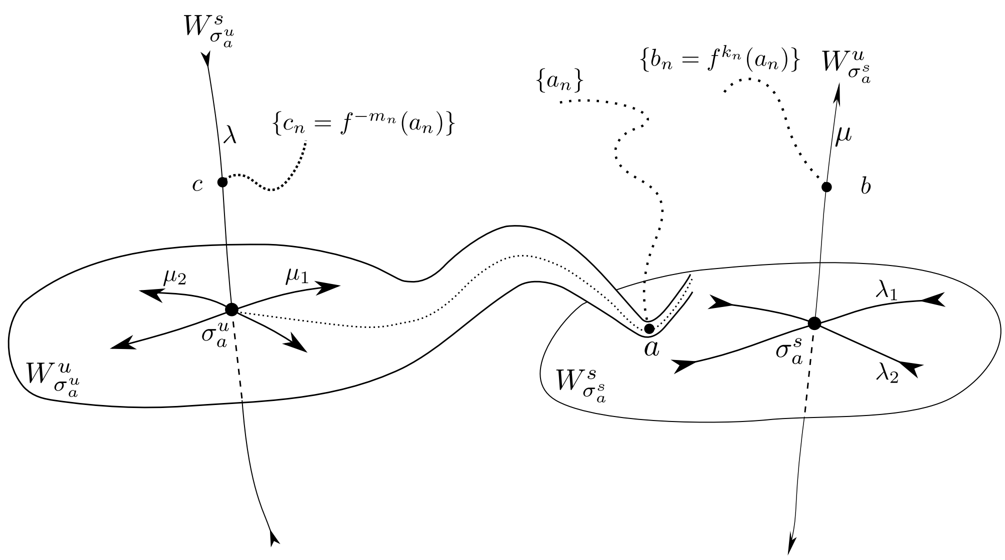

Рис. 2. Illustration for lemma 4

and $\left|\dfrac{[a_\nu]^s_y - [a]^s_y}{[a_\nu]^s_z}\right| < C_2$. Note that these inequalities disallow to take points from $W^s_{\sigma^s_a}$, i.e. $\{a_\nu\} \subset U_a \setminus (W^u_{\sigma^u_a} \cup W^s_{\sigma^s_a})$. From proposition 1 it follows that there exist subsequence $\{a_{\nu_n}\}$, sequences $\{k_n\}$ and $\{m_n\}$, point $b \in (W^u_{\sigma^s_a} \setminus \sigma^s_a)$ and point $c \in (W^s_{\sigma^u_a} \setminus \sigma^u_a)$ such that $\lim\limits_{n\to\infty} k_n = +\infty$, $\lim\limits_{n\to\infty} m_n = +\infty$, $\{b_n = f^{k_n}(a_{\nu_n})\}$ and $\{c_n = f^{-m_n}(a_{\nu_n})\}$ tend to b and c respectively (see fig. 2). For the sake of briefness we denote by $\{a_n\}$ a sequence $\{a_{\nu_n}\}$.

Lemma 4. $\lim\limits_{n\to\infty} \dfrac{m_n}{k_n} = -\dfrac{\ln \mu_a}{\ln \lambda_a}$.

Proof. Since $c_n = f^{-m_n}(a_n)$ and $a_n = f^{-k_n}(b_n)$, it follows that

$$[c_n]^u_z = \lambda_a^{-m_n} \cdot [a_n]^u_z, \quad [b_n]^s_z = \mu_a^{k_n} \cdot [a_n]^s_z.$$

Consider ratio $\dfrac{[c_n]^u_z}{[b_n]^s_z} = \lambda_a^{-m_n} \mu_a^{-k_n} \cdot \dfrac{[a_n]^u_z}{[a_n]^s_z}$. Term $\dfrac{[a_n]^u_z}{[a_n]^s_z}$ can be expressed as

$$\frac{[a_n]^u_z}{[a_n]^s_z} = \frac{\chi_a([a_n]^s_x, [a_n]^s_y, [a_n]^s_z)}{[a_n]^s_z} = \frac{\chi_a([a_n]^s_x, [a_n]^s_y, [a_n]^s_z) - \chi_a([a]^s_x, [a]^s_y, [a]^s_z)}{[a_n]^s_z - [a]^s_z}.$$

Applying mean value theorem, we obtain

$$\frac{\chi_a([a_n]^s_x, [a_n]^s_y, [a_n]^s_z) - \chi_a([a]^s_x, [a]^s_y, [a]^s_z)}{[a_n]^s_z - [a]^s_z} =$$
$$= \frac{\dfrac{\partial\hat{\chi}_a}{\partial z}([a_n]^s_z - [a]^s_z) + \dfrac{\partial\hat{\chi}_a}{\partial x}([a_n]^s_x - [a]^s_x) + \dfrac{\partial\hat{\chi}_a}{\partial y}([a_n]^s_y - [a]^s_y)}{[a_n]^s_z - [a]^s_z} =$$
$$= \frac{\partial\hat{\chi}_a}{\partial z} + \frac{\partial\hat{\chi}_a}{\partial x} \cdot \frac{[a_n]^s_x - [a]^s_x}{[a_n]^s_z - [a]^s_z} + \frac{\partial\hat{\chi}_a}{\partial y} \cdot \frac{[a_n]^s_x - [a]^s_x}{[a_n]^s_z - [a]^s_z},$$

where $\dfrac{\partial\hat\chi_a}{\partial x}$, $\dfrac{\partial\hat\chi_a}{\partial y}$, $\dfrac{\partial\hat\chi_a}{\partial z}$ are the values of corresponding partial derivatives at the intermediate point of segment with a and a_n as an end points. Clearly, the limit of this expression is equal to $\dfrac{\partial\chi_a}{\partial z}(a^s)$ as n tends to infinity; it can be easily shown since $\left|\dfrac{[a_n]^s_x-[a]^s_x}{[a_n]^s_z}\right|$ and $\left|\dfrac{[a_n]^s_y-[a]^s_y}{[a_n]^s_z}\right|$ are bounded and $\dfrac{\partial\chi_a}{\partial x}$, $\dfrac{\partial\chi_a}{\partial y}$ are continuous at point a^s and are equal to zero. Taking logarithm of $\dfrac{[c_n]^u_z}{[b_n]^s_z}$, dividing by $-k_n\cdot\ln\lambda_a$ and rearranging terms, we obtain

$$\frac{m_n}{k_n}+\frac{\ln\mu_a}{\ln\lambda_a}=\frac{1}{k_n\cdot\ln\lambda_a}\cdot\left(\ln\frac{[a_n]^u_z}{[a_n]^s_z}-\ln\frac{[c_n]^u_z}{[b_n]^s_z}\right).$$

Expression in right hand side tends to $\ln\dfrac{\partial\chi_a}{\partial z}(a^s)-\ln\dfrac{[c]^u_z}{[b]^s_z}$ as $n\to+\infty$, so

$$\lim_{n\to\infty}\frac{m_n}{k_n}=-\frac{\ln\mu_a}{\ln\lambda_a}.$$

□

The proof of lemma 5 uses ideas from articles [5] (proof of lemma 2.3) and [9].

By $\ell^{u+}_{\sigma^s_a}$ ($\ell^{u-}_{\sigma^s_a}$) and $\ell^{s+}_{\sigma^u_a}$ ($\ell^{s-}_{\sigma^u_a}$) denote separatrices of invariant manifolds $W^u_{\sigma^s_a}$ and $W^s_{\sigma^u_a}$ such that $\psi_{\sigma^s_a}(\ell^{u+}_{\sigma^s_a})=OZ^+=\{z\in OZ:z>0\}$ ($\psi_{\sigma^s_a}(\ell^{u-}_{\sigma^s_a})=OZ^-=\{z\in OZ:z<0\}$) and $\psi_{\sigma^u_a}(\ell^{s+}_{\sigma^u_a})=OZ^+$ ($\psi_{\sigma^u_a}(\ell^{s-}_{\sigma^u_a})=OZ^-$).

Lemma 5. *Let $a\in\mathcal{A}$ be the point of heteroclinic tangency. Suppose that Θ_a is irrational number. Then for any point $b\in\ell^{u+}_{\sigma^s_a}$ exists $\varepsilon_a\in\{+,-\}$ such that for any point $c\in\ell^{s\varepsilon_a}_{\sigma^u_a}$ exist sequence $\{a_n\}\to a$ and sequences $\{m_n\}\to+\infty$, $\{k_n\}\to+\infty$ such that $\lim\limits_{n\to\infty}f^{k_n}(a_n)=b$, $\lim\limits_{n\to\infty}f^{-m_n}(a_n)=c$.*

Proof. Put $\varepsilon_a=+$ if $\dfrac{\partial\chi_a}{\partial z}(a^s)>0$ and put it equal to $-$ if $\dfrac{\partial\chi_a}{\partial z}(a^s)<0$ For the sake of being definite, take $\varepsilon_a=+$ and $c\in\ell^{s+}_{\sigma^u_a}$. Consider a sequence of points $\{\alpha_m\}$ such that

$$[\alpha_m]^s_x=[a]^s_x,[\alpha_m]^s_y=[a]^s_y,[\alpha_m]^u_z=\lambda^m_a[c]^u_z.$$

Since $c\in\ell^{s+}_{\sigma^u_a}$ then inequality $[\alpha_m]^u_z>0$ holds for any point of sequence $\{\alpha_m\}$. Put by definition $\beta_m=\dfrac{[\alpha_m]^u_z}{[\alpha_m]^s_z}$. From $\dfrac{[\alpha_m]^s_x-[a]^s_x}{[\alpha_m]^s_z}=0$ and $\dfrac{[\alpha_m]^s_y-[a]^s_y}{[\alpha_m]^s_z}=0$ it follows that $\lim\limits_{m\to\infty}\beta_m=\dfrac{\partial\chi_a}{\partial z}(a^s)=\beta_a$ (see lemma 4). Put be definition

$$s_m=\frac{\ln[\alpha_m]^s_z}{\ln\mu_a}=\frac{\ln\left(\dfrac{1}{\beta_m}[\alpha_m]^u_z\right)}{\ln\mu_a}=\frac{\ln\left(\dfrac{1}{\beta_m}\lambda^m_a[c]^u_z\right)}{\ln\mu_a}=\frac{\ln\left([c]^u_z\dfrac{1}{\beta_a}\lambda^m_a\dfrac{\beta_a}{\beta_m}\right)}{\ln\mu_a};$$

then rearranging of terms gives

$$s_m=\frac{\ln\left([c]^u_z\dfrac{1}{\beta_a}\right)}{\ln\mu_a}+\frac{\ln\dfrac{\beta_a}{\beta_m}}{\ln\mu_a}+m\frac{\ln\lambda_a}{\ln\mu_a}.$$

Put by definition $\theta = \dfrac{\ln\left([c]^u_z \dfrac{1}{\beta_a}\right)}{\ln\mu_a}$ and $\zeta_m = \dfrac{\ln\dfrac{\beta_a}{\beta_m}}{\ln\mu_a}$; then $s_m = \theta + \zeta_m + m\dfrac{\ln\lambda_a}{\ln\mu_a}$. Note that $\dfrac{\ln\mu_a}{\ln\lambda_a} = \Theta_a < 0$, $\theta = const$, $\lim\limits_{m\to\infty}\zeta_m = 0$ and $\lim\limits_{m\to\infty} s_m = -\infty$. Consider the mapping $y:\mathbb{R}\to\mathbb{R}$ where $y = x + \omega_a$ and $\omega_a = \dfrac{1}{\Theta_a}$. This map induces a diffeomorphism $\hat{y}:\mathbb{S}^1\to\mathbb{S}^1$ via covering map $p(x) = e^{2\pi i x}$. By construction $\hat{y}$ is a rotation by angle $2\pi\omega_a$ where $\omega_a < 0$ and $\{\theta + m\omega_a\} = \bigcup\limits_{m\in\mathbb{N}} y^m(\theta)$. Since Θ_a is irrational then ω_a is irrational too and $p(\bigcup\limits_{m\in\mathbb{N}} y^m(\theta))$ is dense on circle (see [4], proposition 1.3.3). Then sequence $p(s_m)$ is also dense in circle because $\lim\limits_{m\to\infty}\zeta_m = 0$. For any m number s_m can be expressed as $s_m = \xi_m + \tilde{s}_m$ where ξ_m is an integer part of s_m and $\tilde{s}_m \in [0,1)$. From $\lim\limits_{m\to\infty} s_m = -\infty$ follows that $\lim\limits_{m\to\infty}\xi_m = -\infty$. Since $\{\tilde{s}_m\}$ is dense in $[0,1)$, set $\{\mu_a^{\tilde{s}_m}\}$ is dense in $[1;\mu_a)$. Let q be the integer number such that $\mu_a^q \le [b]^s_z < \mu_a^{q+1}$; then $\mu_a^{q+\tilde{s}_m}$ is dense in $[\mu_a^q, \mu_a^{q+1})$. Hence, for any point $b \in \ell^{u+}_{\sigma^s_a}$, $b = (0,0,[b]^s_z)$ exists subsequence $\{\tilde{s}_{m_n}\}$ such that $[b]^s_z = \mu_a^{\delta+q}$ where $\delta = \lim\limits_{n\to\infty}\tilde{s}_{m_n}$. It now follows that

$$[b]^s_z = \mu_a^q \lim_{n\to\infty}\mu_a^{\tilde{s}_{m_n}} = \mu_a^q \lim_{n\to\infty}\mu_a^{s_{m_n}}\mu_a^{-\xi_{m_n}} = \mu_a^q \lim_{n\to\infty}\mu_a^{-\xi_{m_n}} e^{s_{m_n}\ln\mu_a} =$$
$$= \mu_a^q \lim_{n\to\infty}\mu_a^{-\xi_{m_n}} \exp\left(\frac{\ln[\alpha_{m_n}]^s_z}{\ln\mu_a}\ln\mu_a\right) = \mu_a^q \lim_{n\to\infty}\mu_a^{-\xi_{m_n}}[\alpha_{m_n}]^s_z.$$

Put by definition $-\xi_{m_n} + q = k_n$, $\{a_n\} = \{\alpha_{m_n}\}$, $b_n = f^{k_n}(a_n)$, $c_n = f^{-m_n}(a_n)$. Obviously $\{a_n\}$ is a required sequence. □

Lemma 6. *Let $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear diffeomorphism such that $L(Oz) = Oz$ and $L(Oxy) = Oxy$. Suppose that $L|_{Oz}$ acts like a homothety with coefficient $\mu > 1$ and for any point $P \in Oxy$ iterations $L^n(P)$ tend to O as $n \to +\infty$. Let $\Phi = (\Phi_1(x,y,z), \Phi_2(x,y,z), \Phi_3(x,y,z))$ be a diffeomorphism that commutes with L; also, $\Phi(Oxy) = Oxy$. Then the derivative $\dfrac{\partial\Phi_3}{\partial z}$ is constant at plane Oxy and is non-zero.*

Proof. Since plane Oxy is invariant under map Φ then it is true that $\Phi_3(x,y,0) \equiv 0$ According to Hadamard's lemma (see formulation and proof in [14]), function Φ_3 can be expressed as $z\cdot g(x,y,z)$ where $g(x,y,z)$ is continuous function such that $g(x,y,0) = \dfrac{\partial\Phi_3}{\partial z}\Big|_{(x,y,0)}$. Since maps L and Φ commute then for any $n\in\mathbb{Z}$ maps L^n and Φ commute too. Now consider the sequence of points $\{(x_n,y_n,z_n)\}_{n\in\mathbb{N}}$ such that $x_n = x^*$, $y_n = y^*$, $z_n = \mu^{-2n}$ where x^* and y^* are the coordinates of arbitrary point from plane Oxy. Apply equality $\Phi\circ L^n = L^n\circ\Phi$ to (x_n,y_n,z_n) and calculate the z-coordinate of result. After that we obtain equality

$$\mu^n z_n \cdot g(L^n|_{Oxy}(x_n,y_n), \mu^n z_n) = \mu^n z_n \cdot g(x_n,y_n,z_n),$$

which also can be written as

$$g(L^n|_{Oxy}(x_n,y_n), \mu^n z_n) = g(x_n,y_n,z_n).$$

Passing to the limit, we obtain that $g(0,0,0) = g(x^*, y^*, 0)$, i.e. for any x^* and y^* $\frac{\partial \Phi_3}{\partial z}|_{(x^*,y^*,0)} = \frac{\partial \Phi_3}{\partial z}|_{(0,0,0)}$. From $\Phi_3(x,y,0) \equiv 0$ follows that $\frac{\partial \Phi_3}{\partial x}|_{(x,y,0)} \equiv \frac{\partial \Phi_3}{\partial y}|_{(x,y,0)} \equiv 0$ and $\frac{\partial \Phi_3}{\partial z}|_{(0,0,0)} \neq 0$ because diffeomorphism Φ. has non-zero determinant of Jacobi matrix. □

Lemma 7. *For any points* $d, a \in \mathcal{A}$ *such that* $\sigma_d^s = \sigma_a^s$ *and* $\sigma_d^u = \sigma_a^u$, *parameter* τ_d^a *doesn't depend on choice of linearizing neighbourhoods of saddle point* σ_d^s *and* σ_d^u.

Proof. Recall that $\tau_d^a = \left|\frac{\beta_a}{\beta_d}\right|^{1/\ln \mu_a}$, where $\beta_a = \frac{\partial \chi_a}{\partial z}(a^s)$ and $\beta_d = \frac{\partial \chi_d}{\partial z}(d^s)$ for points $a,\ d \in \mathcal{A}$. It's sufficient to prove that ratio $\frac{\beta_a}{\beta_d}$ doesn't depend on choice of diffeomorphisms $\psi_{\sigma_a^s} : U_{\sigma_a^s} \to U_{J_{\sigma_a^s}}$ and $\psi_{\sigma_a^u} : U_{\sigma_a^u} \to U_{J_{\sigma_a^u}}$.

Recall that for point $a \in \mathcal{A}$ mapping $g_a(x,y,z)$ was defined earlier as

$$g_a = \psi_{\sigma_a^u} \circ (\psi_{\sigma_a^s}|_{U_a})^{-1} : \psi_{\sigma_a^s}(U_a) \to \psi_{\sigma_a^u}(U_a),$$

where U_a is a connected component of $U_{\sigma_a^s} \cap U_{\sigma_a^u}$, which contains point a . Suppose that we've chosen another linearizing neighbourhoods $\tilde{U}_{\sigma_a^s}$, $\tilde{U}_{\sigma_a^u}$ and diffeomorphisms

$$\tilde{\psi}_{\sigma_a^s} : \tilde{U}_{\sigma_a^s} \to U_{J_{\sigma_a^s}}, \quad \tilde{\psi}_{\sigma_a^u} : \tilde{U}_{\sigma_a^u} \to U_{J_{\sigma_a^u}}$$

that don't coincide with $\psi_{\sigma_a^s}$ and $\psi_{\sigma_a^u}$ respectively. By $\tilde{U}_a$ denote the connected component of $\tilde{U}_{\sigma_a^s} \cap \tilde{U}_{\sigma_a^u}$, which contains point a. By definition, put $\tilde{g}_a = \tilde{\psi}_{\sigma_a^u} \circ (\tilde{\psi}_{\sigma_a^s}|_{U_a})^{-1}$; the coordinate expression for $\tilde{g}_a$ will be

$$\tilde{g}_a(x,y,z) = (\tilde{\xi}_a(x,y,z), \tilde{\eta}_a(x,y,z), \tilde{\chi}_a(x,y,z)).$$

Then

$$\tilde{g}_a = \tilde{\psi}_{\sigma_a^u} \circ \psi^{-1}_{\sigma_a^u} \circ \psi_{\sigma_a^u} \circ \psi^{-1}_{\sigma_a^s} \circ \psi_{\sigma_a^s} \circ \tilde{\psi}^{-1}_{\sigma_a^s}.$$

Put by definition $\Psi^s = \tilde{\psi}_{\sigma_a^s} \circ \psi^{-1}_{\sigma_a^s}$ and $\Psi^u = \tilde{\psi}_{\sigma_a^u} \circ \psi^{-1}_{\sigma_a^u}$; after that we obtain $\tilde{g}_a = \Psi^u \circ g_a \circ (\Psi^s)^{-1}$. By construction, diffeomorphisms Ψ^s and Ψ^u commute with linear diffeomorphisms $J_{\sigma_a^s}$ and $J_{\sigma_a^u}$ respectively. Put by definition

$$\Psi^s(x,y,z) = (\Psi_1^s(x,y,z), \Psi_2^s(x,y,z), \Psi_3^s(x,y,z))$$

and

$$\Psi^u(x,y,z) = (\Psi_1^u(x,y,z), \Psi_2^u(x,y,z), \Psi_3^u(x,y,z)).$$

From $\Psi^s(Oxy) = \Psi^u(Oxy) = Oxy$ it follows that $\Psi_3^s(x,y,0) \equiv \Psi_3^u(x,y,0) \equiv 0$; obviously $\frac{\partial \Psi_3^s}{\partial x}(x,y,0) \equiv \frac{\partial \Psi_3^s}{\partial y}(x,y,0) \equiv 0$ and $\frac{\partial \Psi_3^u}{\partial x}(x,y,0) \equiv \frac{\partial \Psi_3^u}{\partial y}(x,y,0) \equiv 0$. Note that from $\tilde{g}_a = \Psi^u \circ g_a \circ (\Psi^s)^{-1}$ follows that

$$\mathrm{D}\tilde{g}_a\Big|_{([\tilde{a}]^s_x, [\tilde{a}]^s_y, 0)} = \mathrm{D}\Psi^u\Big|_{([a]^u_x, [a]^u_y, 0)} \cdot \mathrm{D}g_a\Big|_{([a]^s_x, [a]^s_y, 0)} \cdot \mathrm{D}(\Psi^s)^{-1}\Big|_{([\tilde{a}]^s_x, [\tilde{a}]^s_y, 0)},$$

where Jacobi matrices are taken at point $a \in \mathcal{A}$. According to lemmas 3 and 6, Jacobi matrices have following form:

$$\mathrm{D}g_a\Big|_{([a]^s_x,[a]^s_y,0)} = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & \dfrac{\partial \chi_a}{\partial z}(a^s) \end{pmatrix}, \quad \mathrm{D}\Psi^u\Big|_{([a^u_x],[a^u_y],0)} = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & \dfrac{\partial \Psi^u_3}{\partial z}(a^u) \end{pmatrix},$$

$$\mathrm{D}(\Psi^s)^{-1}\Big|_{([\tilde{a}^s_x],[\tilde{a}^s_y],0)} = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & \left(\dfrac{\partial \Psi^s_3}{\partial z}\right)^{-1}(\tilde{a}^s) \end{pmatrix},$$

where star signs denote coefficients that are irrelevant to the proof. Multiplying Jacobi matrices, we get equality

$$\frac{\partial \tilde{\chi}_a}{\partial z}\Big|_{([\tilde{a}]^s_x,[\tilde{a}]^s_y,0)} = \frac{\partial \Psi^u_3}{\partial z}\Big|_{([a]^u_x,[a]^u_y,0)} \cdot \frac{\partial \chi_p}{\partial z}\Big|_{([a]^s_x,[a]^s_y,0)} \cdot \left(\frac{\partial \Psi^3_s}{\partial z}\right)^{-1}\Big|_{([\tilde{a}]^s_x,[\tilde{a}]^s_y,0)}$$

which can be combined with lemma 6 and rewritten as

$$\frac{\partial \tilde{\chi}_p}{\partial z}\Big|_{([\tilde{a}]^s_x,[\tilde{a}]^s_y,0)} = \frac{\partial \Psi^u_3}{\partial z}\Big|_{(0,0,0)} \cdot \frac{\partial \chi_p}{\partial z}\Big|_{([a]^s_x,[a]^s_y,0)} \cdot \left(\frac{\partial \Psi^3_s}{\partial z}\right)^{-1}\Big|_{(0,0,0)}.$$

The same formula holds for point $d \in \mathcal{A}$. This means that

$$\frac{\tilde{\beta}_a}{\tilde{\beta}_d} = \frac{\dfrac{\partial \tilde{\chi}_a}{\partial z}(\tilde{a}^s)}{\dfrac{\partial \tilde{\chi}_d}{\partial z}(\tilde{d}^s)} = \frac{\dfrac{\partial \Psi^u_3}{\partial z}(0,0,0)\dfrac{\partial \chi_a}{\partial z}(a^s)\dfrac{\partial \Psi^s_3}{\partial z}(0,0,0)}{\dfrac{\partial \Psi^u_3}{\partial z}(0,0,0)\dfrac{\partial \chi_d}{\partial z}(d^s)\dfrac{\partial \Psi^s_3}{\partial z}(0,0,0)} = \frac{\dfrac{\partial \chi_a}{\partial z}(a^s)}{\dfrac{\partial \chi_d}{\partial z}(d^s)} = \frac{\beta_a}{\beta_d}.$$

□

Suppose that $\hat{U}_{a^s}$ is some euclidean neighbourhood of tangency point $a^s \in U_{J_{\sigma^s_a}}$, $\hat{U}_a = \psi^{-1}_{\sigma^s_a}(\hat{U}_{a^s}) \subset U_a$. Suppose that partial derivative $\dfrac{\partial \chi_a}{\partial z}$ doesn't change sign in $\hat{U}_{a^s}$ (the existence of such neighbourhood follows from continuity of partial derivative). Also suppose that $\psi_{\sigma^s_a}(W^u_{\sigma^u_a} \cap \hat{U}_a)$ intersects exactly one connected component of $\hat{U}_{a^s} \setminus \psi_{\sigma^s_a}(W^s_{\sigma^s_a} \cap \hat{U}_a)$ (this is possible because of one-sidedness of tangency). By $\hat{U}^+_{a^s}$ and $\hat{U}^-_{a^s}$ denote sets $\{p \in \hat{U}_{a^s} \colon [p]^s_z > 0\}$ and $\{p \in \hat{U}_{a^s} \colon [p]^s_z < 0\}$ respectively. Also denote by ε_a the sign of partial derivative $\dfrac{\partial \chi_a}{\partial z}(a^s)$; sign that is opposite to ε_a we will denote by $\bar{\varepsilon}_a$. Let a and a' be the points of heteroclinic tangency, $h(a) = a'$. Suppose that for neighbourhood $U_{\sigma^s_a}$ holds $h(U_{\sigma^s_a}) \subseteq U_{\sigma^s_{a'}}$. It's always possible to choose such linearizing neighbourhood. Suppose that $h(U_{\sigma^s_a}) \not\subseteq U_{\sigma^s_{a'}}$; then there exists $k \in \mathbb{N}$ such that $h(U^k_{\sigma^s_a}) \subseteq U_{\sigma^s_{a'}}$, where $U^k_{\sigma^s_a} = \psi^{-1}_{\sigma^s_a}(U^k_{J_{\sigma^s_a}})$ (this observation is similar to the proof of lemma 2). So, linearizing neighbourhood $U^k_{\sigma^s_a}$ satisfies this condition. Then we can define the homeomorphism $\hat{h}_s \colon \psi_{\sigma^s_a}(U_{\sigma^s_a}) \to \psi_{\sigma^s_{a'}}(h(U_{\sigma^s_a}))$ by formula $\hat{h}_s = \psi_{\sigma^s_{a'}} h \psi^{-1}_{\sigma^s_a}$. Point a'^s is an image of point a^s under the mapping $\hat{h}_s$; put by definition $\hat{U}_{a'^s} = \hat{h}_s(\hat{U}_{a^s})$. For neighbourhood $\hat{U}_{a'^s}$ we define sets $\hat{U}^+_{a'^s}$ and $\hat{U}^-_{a'^s}$ in similar manner as for $\hat{U}_{a^s}$. Note that analogous constructions can be made to define a homeomorphism $\hat{h}_u \colon \psi_{\sigma^u_a}(U_{\sigma^u_a}) \to \psi_{\sigma^u_{a'}}(h(U_{\sigma^u_a}))$.

Lemma 8. *If* $p^s \in \hat{U}^{\varepsilon_a}_{a^s}$ *then* $\chi_a([p]^s_x, [p]^s_y, [p]^s_z) > \chi_a([p]^s_x, [p]^s_y, 0)$ *and if* $p^s \in \hat{U}^{\bar{\varepsilon}_a}_{a^s}$ *then* $\chi_a([p]^s_x, [p]^s_y, [p]^s_z) < \chi_a([p]^s_x, [p]^s_y, 0)$.

Proof. Statement can be proven via considering the expression $\chi_a([p]^s_x, [p]^s_y, [p]^s_z) - \chi_a([p]^s_x, [p]^s_y, 0)$. Applying the mean value theorem, we obtain

$$\chi_a([p]^s_x, [p]^s_y, [p]^s_z) - \chi_a([p]^s_x, [p]^s_y, 0) = [p]^s_z \cdot \frac{\partial \hat{\chi}_a}{\partial z},$$

where $\frac{\partial \hat{\chi}_a}{\partial z}$ is a value of partial derivative $\frac{\partial \chi_a}{\partial z}$ at some intermediate point of segment with $([p]^s_x, [p]^s_y, [p]^s_z)$ and $([p]^s_x, [p]^s_y, 0)$ as an endpointes. Since the sign of $\frac{\partial \hat{\chi}_a}{\partial z}$ coincides with ε_a in neighbourhood $\hat{U}_{a^s}$, the sign of $\chi_a([p]^s_x, [p]^s_y, [p]^s_z) - \chi_a([p]^s_x, [p]^s_y, 0)$ is equal to $\varepsilon_a \cdot \mathrm{sgn}[p]^s_z$. $\square$

4. Necessary conditions for topological conjugacy

Theorem 1. *Suppose that* $f, f' \in \Psi$ *are topologically conjugated via homeomorphism* h *such that* $h(a) = a'$ *for point* $a \in \mathcal{A}$, $h(\sigma^s_a) = \sigma^s_{a'}$, $h(\sigma^u_a) = \sigma^u_{a'}$. *Then* $\Theta_a = \Theta_{a'}$.

Proof. We will mark with stroke sign all objects of diffeomorphism f' that are images of corresponding objects of diffeomorphism f under homeomorphism h.

First, we choose linearizing neighbourhood $U_{\sigma^u_{a'}}$ as it was described before lemma 8. After that, we choose the mapping $\psi_{\sigma^u_{a'}}$ such that images of points of $W^s_{\sigma^s_{a'}}$ under $\psi_{\sigma^u_{a'}}$ has non-negative z-coordinate in some neighbourhood of tangency point a'^u (in the opposite case we can apply changing of coordinates $\mathrm{mir}_z : (x, y, z) \to (x, y, -z)$ and set $\tilde{\psi}_{\sigma^u_{a'}} = \mathrm{mir}_z \circ \psi_{\sigma^u_{a'}}$). Thus it is possible to chose sequence $\{a_n\}$ befor lemma 4 such that for $a'_n = h(a_n)$ the following relations hold

$$\chi_{a'}([a'_n]^s_x, [a'_n]^s_y, [a'_n]^s_z) > \chi_{a'}([a'_n]^s_x, [a'_n]^s_y, 0) \geq 0.$$

As a result of choice we have sequence of points $\{a_n\}$, integer sequences $\{k_n\}$ and $\{m_n\}$, points $b \in (W^u_{\sigma^s_a} \setminus \sigma^s_a)$ and $c \in (W^s_{\sigma^u_a} \setminus \sigma^u_a)$ such that $\lim\limits_{n\to\infty} k_n = +\infty$, $\lim\limits_{n\to\infty} m_n = +\infty$ and sequences $\{b_n = f^{k_n}(a_n)\}$, $\{c_n = f^{-m_n}(a_n)\}$ tend to points b and c respectively; moreover,

$$\chi_{a'}([a'_n]^s_x, [a'_n]^s_y, [a'_n]^s_z) > \chi_{a'}([a'_n]^s_x, [a'_n]^s_y, 0) \geq 0.$$

There are two possibilities here:

1) *Sequence* $\{a'_n\}$ *has subsequence* $\{a'_{n_q}\}$ *such that exist positive constants* C_1 *and* C_2 *and the inequalities* $\left|\frac{[a'_{n_q}]^s_x - [a']^s_x}{[a'_{n_q}]^s_z}\right| < C_1$ *and* $\left|\frac{[a'_{n_q}]^s_y - [a']^s_y}{[a'_{n_q}]^s_z}\right| < C_2$ *hold for all elements of subsequence.*

   In this case both sequences $\{a_{n_q}\}$ and $\{a'_{n_q}\}$ satisfy conditions of lemma 4. On one hand, $\lim\limits_{q\to\infty} \frac{m_{n_q}}{k_{n_q}} = -\frac{\ln \mu_a}{\ln \lambda_a}$; on the other hand, $\lim\limits_{q\to\infty} \frac{m_{n_q}}{k_{n_q}} = -\frac{\ln \mu_{a'}}{\ln \lambda_{a'}}$. From that we obtain that $\frac{\ln \mu_a}{\ln \lambda_a} = \frac{\ln \mu_{a'}}{\ln \lambda_{a'}}$.

2) *Sequence* $\{a'_n\}$ *has no subsequence, which satisfies conditions of case* 1).

From conditions for linearizing neighbourhoods and sequence $\{a_n\}$ follows that

$$\chi_{a'}([a'_n]^s_x, [a'_n]^s_y, [a']^s_z) > 0.$$

For images of sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ following equalities hold:

$$[a'_n]^s_z = \mu_{a'}^{-k_n} \cdot [b'_n]^s_z, \ [c'_n]^u_z = \lambda_{a'}^{-m_n} \cdot [a'_n]^u_z,$$

$$[a'_n]^u_z - \chi_{a'}([a'_n]^s_x, [a'_n]^s_y, [a']^s_z) = B_n \cdot \mu_{a'}^{-k_n} \cdot [b'_n]^s_z,$$

where

$$B_n = \frac{[a'_n]^u_z - \chi_{a'}([a'_n]^s_x, [a'_n]^s_y, [a']^s_z)}{[a'_n]^s_z}.$$

Since

$$\chi_{a'}([a'_n]^s_x, [a'_n]^s_y, [a']^s_z) > 0,$$

then

$$[a'_n]^u_z - \chi_{a'}([a'_n]^s_x, [a'_n]^s_y, [a']^s_z) < [a'_n]^u_z.$$

From last inequality follows that

$$B_n \cdot \mu_{a'}^{-k_n} \cdot [b'_n]^s_z < [a'_n]^u_z.$$

Multiplicating by $\lambda_{a'}^{-m_n}$ and dividing by $B_n \cdot [b'_n]^s_z$, we get

$$\lambda_{a'}^{-m_n} \mu_{a'}^{-k_n} < \frac{[c'_n]^u_z}{B_n \cdot [b'_n]^s_z}.$$

After taking logarithm and dividing by $-k_n \cdot \ln \lambda_{a'}$ we obtain

$$\frac{m_n}{k_n} > -\frac{\ln \mu_{a'}}{\ln \lambda_{a'}} - \frac{1}{k_n} \cdot \frac{\ln\left(\dfrac{[c'_n]^u_z}{B_n \cdot [b'_n]^s_z}\right)}{\ln \lambda_{a'}}.$$

Obviously, we have $\lim\limits_{n\to\infty} \dfrac{1}{k_n} = 0$, $\lim\limits_{n\to\infty} [c'_n]^u_z = [c']^u_z$, $\lim\limits_{n\to\infty} [b'_n]^s_z = [b']^s_z$ and $B_n[b'_n]^s_z > 0$. Also we have that $\lim\limits_{n\to\infty} B_n = \dfrac{\partial \chi_{a'}}{\partial x}(a'^s)$ because

$$B_n = \frac{\chi_{a'}([a'_n]^s_x, [a'_n]^s_y, [a'_n]^s_z) - \chi_{a'}([a'_n]^s_x, [a'_n]^s_y, [a']^s_z)}{[a'_n]^s_z - [a']^s_z} =$$

$$= \frac{\dfrac{\partial \hat\chi_{a'}}{\partial z}([a'_n]^s_z - [a']^s_z) + \dfrac{\partial \hat\chi_{a'}}{\partial x}([a'_n]^s_x - [a'_n]^s_x) + \dfrac{\partial \hat\chi_{a'}}{\partial y}([a'_n]^s_y - [a'_n]^s_y)}{[a'_n]^s_z - [a']^s_z} = \frac{\partial \hat\chi_{a'}}{\partial z},$$

where $\dfrac{\partial \hat\chi_{a'}}{\partial x}$, $\dfrac{\partial \hat\chi_{a'}}{\partial y}$, $\dfrac{\partial \hat\chi_{a'}}{\partial z}$ are the values of corresponding partial derivatives at some intermediate point of segment with a'^s and a'^s_n as an endpointes. From all this we conclude that $\lim\limits_{n\to\infty}\dfrac{m_n}{k_n} \geq -\dfrac{\ln\mu_{a'}}{\ln\lambda_{a'}}$. From lemma 4 follows $\lim\limits_{n\to\infty}\dfrac{m_n}{k_n} = -\dfrac{\ln\mu_a}{\ln\lambda_a}$, so $\dfrac{\ln\mu_a}{\ln\lambda_a} \leq \dfrac{\ln\mu_{a'}}{\ln\lambda_{a'}}$.

If we begin with diffeomorphism f', we can obtain that $\lim\limits_{n\to\infty}\dfrac{m'_n}{k'_n} = -\dfrac{\ln\mu_{a'}}{\ln\lambda_{a'}}$ and $\lim\limits_{n\to\infty}\dfrac{m'_n}{k'_n} \geq -\dfrac{\ln\mu_a}{\ln\lambda_a}$. From this follows that $\dfrac{\ln\mu_{a'}}{\ln\lambda_{a'}} \leq \dfrac{\ln\mu_a}{\ln\lambda_a}$. Obviously, $\dfrac{\ln\mu_a}{\ln\lambda_a} = \dfrac{\ln\mu_{a'}}{\ln\lambda_{a'}}$.

$\square$

Let a be an arbitrary point of heteroclinic tangency. Suppose that linearizing neighbourhoods $U_{\sigma_a^s}$ and $U_{\sigma_a^u}$ are such that homeomorphisms $\hat{h}_s$ and $\hat{h}_u$ can be defined as in description before lemma 8. Denote by $\hat{H}_s$ and $\hat{H}_u$ restrictions $\hat{h}_s\big|_{Oz}$ and $\hat{h}_u\big|_{Oz}$. Also, suppose that neighbourhoods $\hat{U}_{a^s}$ and $\hat{U}_{a'^s} = \hat{h}_s(\hat{U}_{a^s})$ are defined as before lemma 8, but with one additional condition: sign of $\dfrac{\partial\chi_{a'}}{\partial z}$ in $\hat{U}_{a'^s}$ coincides with sign of $\dfrac{\partial\chi_{a'}}{\partial z}(a'^s)$. This condition always can be treated by choosing smaller euclidean neighbourhood inside $\hat{U}_{a^s}$.

Lemma 9. *Let diffeomorphisms $f, f' \in \Psi^*$ be conjugated by a homeomorphism h. Let $a \in \mathcal{A}$ be an arbitrary point of heteroclinic tangency and $h(a) = a'$. Then induced conjugating homeomorphisms $\hat{H}_s$ and $\hat{H}_u$ have following coordinate expression*

$$\hat{H}_s(z) = \begin{cases} \alpha_s^+ \cdot z^\rho, & z > 0 \\ \alpha_s^- \cdot (-z)^\rho, & z < 0 \end{cases} \text{ and } \hat{H}_u(z) = \begin{cases} \alpha_u^+ \cdot z^\rho, & z > 0 \\ \alpha_u^- \cdot (-z)^\rho, & z < 0 \end{cases}, \text{ where } \rho = \frac{\ln\mu_{a'}}{\ln\mu_a} = \frac{\ln\lambda_{a'}}{\ln\lambda_a}.$$

Proof. Take any tangency point $a \in \mathcal{A}$ and corresponding saddle fixed points σ_a^s and σ_a^u. Homeomorphism h maps point a to point a'; also, h maps saddle fixed points σ_a^s and σ_a^u to $\sigma_{a'}^s$ and $\sigma_{a'}^u$ respectively.

Using approach that we've mentioned in proof of theorem 1, we modify mappings $\psi_{\sigma_{a'}^u}$, $\psi_{\sigma_a^u}$, $\psi_{\sigma_a^s}$ и $\psi_{\sigma_{a'}^s}$. Choose $\psi_{\sigma_{a'}^u}$ and $\psi_{\sigma_a^u}$ such that images of points of invariant manifolds $W^s_{\sigma_{a'}^s}$ и $W^s_{\sigma_a^s}$ under $\psi_{\sigma_{a'}^u}$ and $\psi_{\sigma_a^u}$ have non-negative z-coordinate in some neighbourhoods of tangency points a'^u and a^u respectively. Also, choose $\psi_{\sigma_a^s}$ such that for all points $p^s \in \hat{U}^+_{\sigma_a^s}$ holds $\chi_a(p^s) > \chi_a([p]^s_x, [p]^s_y, 0) \geq 0$. Similarly choose $\psi_{\sigma_{a'}^s}$ such that for all points $p'^s \in \hat{U}^+_{\sigma_{a'}^s}$ holds $\chi_a(p'^s) > \chi_a([p']^s_x, [p']^s_y, 0) \geq 0$. Note that as a consequence of this choice of neighbourhoods and mappings we have that partial derivatives $\dfrac{\partial\chi_a}{\partial z}(a^s)$ and $\dfrac{\partial\chi_{a'}}{\partial z}(a'^s)$ are positive; also, for homeomorphisms $\hat{H}_s$ and $\hat{H}_u$ following holds:

$$\hat{H}_s\colon OZ^+ \subset U_{J_{\sigma_a^s}} \to OZ^+ \subset U_{J_{\sigma_{a'}^s}},\ \hat{H}_s\colon OZ^- \subset U_{J_{\sigma_a^s}} \to OZ^- \subset U_{J_{\sigma_{a'}^s}},$$

$$\hat{H}_u\colon OZ^+ \subset U_{J_{\sigma_a^u}} \to OZ^+ \subset U_{J_{\sigma_{a'}^u}},\ \hat{H}_u\colon OZ^- \subset U_{J_{\sigma_a^u}} \to OZ^- \subset U_{J_{\sigma_{a'}^u}};$$

in other words, this means that $\alpha_s^+,\ \alpha_u^+ > 0$ and $\alpha_s^-,\ \alpha_u^- < 0$.

Applying lemma 5, we obtain that for any point $c \in \ell^{s+}_{\sigma_a^u}$ exist sequence $\{a_n\} \to a$, $\{a_n\} \subset (U_a \setminus (W^s_{\sigma_a^s} \cup W^u_{\sigma_a^u}))$ and integer sequences $\{k_n\} \to +\infty$, $\{m_n\} \to +\infty$ such that $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} f^{k_n}(a_n) = b$ (moreover, $b \in \ell^{u+}_{\sigma_a^s}$) and $\lim\limits_{n\to\infty} c_n = \lim\limits_{n\to\infty} f^{-m_n}(a_n) = c$. By construction, we get that $[b_n]^s_z = \mu_a^{k_n}\dfrac{1}{\beta_n}\lambda_a^{m_n}[c]^u_z$, where $\beta_n = \dfrac{[a_n]^u_z}{[a_n]^s_z}$; then, $\mu_a^{k_n}\lambda_a^{m_n} = \dfrac{[b_n]^s_z\beta_n}{[c]^u_z}$. From $\lim\limits_{n\to\infty}[b_n]^s_z = [b]^s_z$ and $\lim\limits_{n\to\infty}\beta_n = \beta_a$, follows $\lim\limits_{n\to\infty}\mu_a^{k_n}\lambda_a^{m_n} = \dfrac{[b]^s_z\beta_a}{[c]^u_z}$.

We will mark with stroke sign all objects of diffeomorphism f' that are images of corresponding objects of diffeomorphism f under conjugating homeomorphism h. For diffeomorphism f' we have similar formulas $\mu_{a'}^{k_n}\lambda_{a'}^{m_n} = \dfrac{[b'_n]^s_z\beta'_n}{[c']^u_z}$, где $\beta'_n = \dfrac{[a'_n]^u_z}{[a'_n]^s_z}$. According to theorem 1 $\Theta_a = \Theta_{a'}$, i.e. $\dfrac{\ln\mu_a}{\ln\lambda_a} = \dfrac{\ln\mu_{a'}}{\ln\lambda_{a'}}$. Put by definition $\rho = \dfrac{\ln\mu_{a'}}{\ln\mu_a} = \dfrac{\ln\lambda_{a'}}{\ln\lambda_a}$. Obviously, $\mu_{a'}^{k_n}\lambda_{a'}^{m_n} = (\mu_a^{k_n}\lambda_a^{m_n})^\rho$ and $\lim\limits_{n\to\infty}\mu_{a'}^{k_n}\lambda_{a'}^{m_n} = \left(\dfrac{[b]^s_z\beta_a}{[c]^u_z}\right)^\rho$. Now we obtain

$$\left(\frac{[b_n]^s_z\beta_n}{[c_n]^u_z}\right)^\rho = (\mu_a^{k_n}\lambda_a^{m_n})^\rho = \mu_{a'}^{k_n}\lambda_{a'}^{m_n} = \frac{[b'_n]^s_z\beta'_n}{[c'_n]^u_z} = \frac{[b'_n]^s_z[a'_n]^u_z}{[c'_n]^u_z[a'_n]^s_z}$$

and

$$\frac{[b'_n]^s_z[a'_n]^u_z}{[c'_n]^u_z[a'_n]^s_z} \geq \frac{[b'_n]^s_z\left([a'_n]^u_z - \chi_{a'}\left([a'_n]^s_x,[a'_n]^s_y,0\right)\right)}{[c'_n]^u_z[a'_n]^s_z}.$$

Applying similar reasoning as in proof of lemma 4, we conclude that

$$\frac{[a'_n]^u_z - \chi_{a'}([a'_n]^s_x,[a'_n]^s_y,0)}{[a'_n]^s_z}$$

tends to $\beta_{a'}$ as $n\to\infty$. Passing to the limit, we get

$$\left(\frac{[b]^s_z\beta_a}{[c]^u_z}\right)^\rho \geq \frac{[b']^s_z\beta_{a'}}{[c']^u_z}.$$

If we start from diffeomorphism f' and apply similar considerations, we'll get that

$$\left(\frac{[b']^s_z\beta_{a'}}{[c']^u_z}\right)^{\dfrac{1}{\rho}} \geq \frac{[b]^s_z\beta_a}{[c]^u_z}.$$

Hence,

$$\left(\frac{[b]^s_z\beta_a}{[c]^u_z}\right)^\rho = \frac{[b']^s_z\beta_{a'}}{[c']^u_z};$$

in other words,

$$\frac{|\beta_a|^\rho}{|\beta_{a'}|} = \frac{|[b']^s_z|\cdot|[c]^u_z|^\rho}{|[c']^u_z|\cdot|[b]^s_z|^\rho}.$$

Let's interpret last formula. If we fix point c and vary point b arbitrarily, then it follows that $\dfrac{|[b']^s_z|}{|[b]^s_z|^\rho} = \mathrm{const}$; similarly, if we fix point b and vary point c, we get $\dfrac{|[c]^u_z|^\rho}{|[c']^u_z|} = \mathrm{const}$. From this follows that $[b']^s_z = \alpha_s^+([b]^s_z)^\rho$ and $[c']^u_z = \alpha_u^+([c]^u_z)^\rho$; these formulas define homeomorphisms $\hat{H}_s^+\colon OZ^+\to OZ^+$ and $\hat{H}_u^+\colon OZ^+\to OZ^+$. If we take point $c\in\ell^{s-}_{\sigma_a^u}$, we can prove similar formula for homeomorphisms $\hat{H}_s^-\colon OZ^-\to OZ^-$ and $\hat{H}_u^-\colon OZ^-\to OZ^-$, namely $[b']^s_z = \alpha^+_s(-[b]^s_z)^\rho$ and $[c']^u_z = \alpha^+_u(-[c]^u_z)^\rho$ respectively. In terms of induced homeomorphisms we can write formula as

$$\frac{|\beta_a|^\rho}{|\beta_{a'}|} = \frac{|\alpha^+_s|}{|\alpha^+_u|} = \frac{|\alpha^-_s|}{|\alpha^-_u|}.$$

Note that proof for this lemma was given in particular case of mappings $\psi_{\sigma^u_{a'}}$, $\psi_{\sigma^u_a}$, $\psi_{\sigma^s_a}$ and $\psi_{\sigma^s_{a'}}$. However, all modifications that we've applied are just compositions of mirror symmetries mir_z with original mappings. We can revert these modifications, substitute current coordinates with "old" coordinates and obtain similar formulas for $\hat{H}_s$ and $\hat{H}_u$ for all cases. □

Recall that in theorem 2 we consider tangency points $a,\ d \in \mathcal{A}$ such that $\sigma^s_d = \sigma^s_a$, $\sigma^u_d = \sigma^u_a$ and signs of parameters β_d, β_a coincide.

Theorem 2. *Suppose that $f, f' \in \Psi^*$ are topologically conjugated via homeomorphism h such that $h(a) = a'$, $h(d) = d'$ for points $a, d \in \mathcal{A}$ such that $\beta_a \cdot \beta_d > 0$, $h(\sigma^s_a) = \sigma^s_{a'}$ and $h(\sigma^u_a) = \sigma^u_{a'}$. Then $\tau^a_d = \tau^{a'}_{d'}$.*

Proof. Take any of points a or d (for example, a) and choose linearizing neighbourhoods similarly as in proof of lemma 9. From lemma 7 follows that coincidence of signs of parameters β_d and β_a doesn't depend on choice of linearizing neighbourhoods. It's not hard to show that procedure of choice from lemma 9 entails that signs of $\beta_{d'}$ and $\beta_{a'}$ coincide too. But this leads to $\dfrac{|\beta_a|^\rho}{|\beta_{a'}|} = \dfrac{|\alpha^+_s|}{|\alpha^+_u|}$ and $\dfrac{|\beta_d|^\rho}{|\beta_{d'}|} = \dfrac{|\alpha^+_s|}{|\alpha^+_u|}$. From this follows that $\dfrac{|\beta_a|^\rho}{|\beta_{a'}|} = \dfrac{|\beta_d|^\rho}{|\beta_{d'}|}$; then, $\left|\dfrac{\beta_a}{\beta_d}\right|^{\frac{1}{\ln \mu_a}} = \left|\dfrac{\beta_{a'}}{\beta_{d'}}\right|^{\frac{1}{\ln \mu_{a'}}}$, i.e. coincidence of parameters $\tau^a_d = \tau^{a'}_{d'}$. □

References

1. *V*.I. Arnold, V.S. Afraimovich, Yu.S. Il'yashenko, L.P. Shil'nikov, Theory of bifurcations of dynamical systems, Bifurcation Theory, Dynamical Systems, Encyclopedia of Mathematical Sciences 5, 1989.
2. *G*.R. Belitskii, V.A. Tkachenko, Normal Forms, Invariants, and Local Mappings, Kyiv: Naukova dumka (1979).
3. *V*.Z. Grines, O.V. Pochinka, Vvedenie v topologicheskuyu klassifikaciyu kaskadov na 2- i 3-mnogoobraziyah (Introduction to a topological classification of cascades on 2- and 3-manifolds)(in Russian), Regular and Chaotic Dynamics, Institute of Computer Science, Izhevsk, 2011.
4. *A*.B. Katok, B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, Cambridge University Press, 1995.
5. *W*. Melo, Moduli of stability of two-dimensional diffeomorphisms, Topology 19 (1980) 9–21.
6. *W*. Melo, J. Palis, Moduli of stability for diffeomorphisms, Proc. Symp. Dynamical Systems Northwestern Springer Lectures Notes 819 (1980) 318–339.
7. *W*. Melo, J. Palis, Geometric theory of dynamical systems, An introduction, Springer-Verlag, New York-Berlin, 1982.
8. *W*. Melo, S.J. Strien, Diffeomorphisms on surfaces with a finite number of moduli, Ergod. Th. and Dynam. Sys. 7 (1987) 415–462.
9. *T*.M. Mitryakova, O.V. Pochinka, O klassifikacii diffeomorfizmov poverhnostei s konechnym chislom modulei topologicheskoi sopryazhennosti (On classification of diffeomorphisms of surfaces with a finite number of moduli of topological conjugacy), Nelinejnaya dinamika 6 (1) (2010) 91–105.

10. *T*.M. Mitryakova, O.V. Pochinka, E.A. Shishenkova, O strukture prostranstva bluzhdayushih orbit diffeomorfizmov poverhnostei s konechnym giperbolicheskim cepno rekurentnym mnozhestvom (About a structure of space of nonwandering orbits of surface diffeomorphisms with finite hyperbolic chain recurrent set), Journal SVMO 13 (1) (2011) 63–70.
11. *S*. Newhouse, J. Palis, Bifurcations of Morse-Smale dynamical systems, In M. M. Peixoto, editor, Dynamical Systems: Proc. Symp. Bahia, Brazil 14 (1971) 303–366.
12. *S*. Newhouse, J. Palis, F. Takens, Bifurcations and stability of families of diffeomorphisms, Publications Mathématiques de l'Institut des Hautes Études Scientifiques 57 (1) (1983) 5–71.
13. *J*. Palis, A differentiable invariant of topological conjugacies and moduli of stability, Asterisque 51 (1) (1978) 335–346.
14. *I*.G. Petrovskii, Lekcii po teorii obyknovennyh differencial'nyh uravnenii (Lectures on the theory of ordinary differential equations), Moscow: Izdat. Moskovskogo Univ., 1984.
15. *L*.P. Shilnikov, A.L. Shilnikov, D.V. Turaev, L.O. Chua, Methods of qualitative theory in nonlinear dynamics. Part I. World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises, 4. World Scientific Publishing Co., Inc., River Edge, NJ (1998).

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